Vieta on Cyclotomic Polynomials

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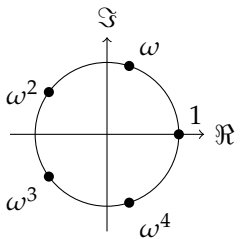
The key idea: a product over roots of unity is a polynomial evaluated at a point — write the polynomial, plug in.



the 5th roots of unity, $\omega = e^{2\pi i/5}$

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The Model Product

A product over roots of unity is a polynomial evaluated at a point — write the polynomial, plug in.

The technique in two sentences

When to reach for it: you see a product that runs over roots of unity — like $\prod(1-\omega^k)$, or a product of all the differences $\omega^i - \omega^j$.

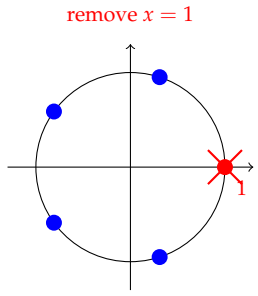
The move: name the polynomial whose roots are exactly those roots of unity, write it in factored form $\prod(x-\omega^k)$, and plug in the point that turns each factor into your factor.

Why it works: a monic polynomial *is* the product of (x minus each root) — the factored form behind Vieta. So a messy product of complex numbers becomes one plug-in of a polynomial with integer coefficients.

The model product: setup

The n-th roots of unity are the roots of $x^n - 1$.
Removing $x = 1$ leaves, with $\omega = e^{2\pi i/n}$,

$$x^{n-1} + \cdots + x + 1 = \prod_{i=1}^{n-1} (x - \omega^i).$$



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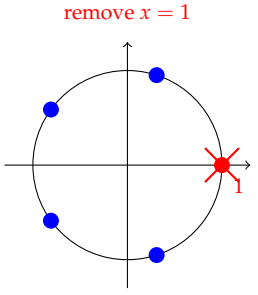
The n-th roots of unity are the roots of $x^n - 1$. Removing $x = 1$ leaves, with $\omega = e^{2\pi i/n}$,

$$x^{n-1} + \cdots + x + 1 = \prod_{i=1}^{n-1} (x - \omega^i).$$

Plug in the evaluation point: at $x = 1$ each factor becomes $1 - \omega^i$, and the left side is n ones:

$$\prod_{i=1}^{n-1} (1 - \omega^i) = \boxed{n}.$$



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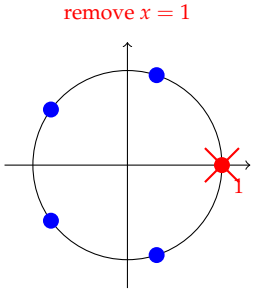
Cue: when you see a product over roots of unity, write the polynomial whose roots they are and plug in the evaluation point.



Factoring Out the Common Power

Factor the common power out first; the cyclotomic product does the rest.

Cyclotomic products: Example

Let $\omega = e^{2\pi i/5}$. Compute

$$\prod_{1 \le i < j \le 5} (\omega^i - \omega^j)^2.$$

Strategy: trade the 10 squared differences for the 20 ordered differences (the signs cancel in pairs); fix j and factor the common power ω^j out of each term. [1]

[1] BMT 2025/7

Cyclotomic products: Try it

Let $\omega = e^{2\pi i/5}$. Compute

$$\prod_{1 \le i < j \le 5} (\omega^i - \omega^j)^2.$$

Hint: $\omega^j - \omega^i = \omega^j(1 - \omega^{i-j})$. As i runs over $\{1, \ldots, 5\} \setminus \{j\}$, what does $i - j$ run through mod 5? [2]

[2] BMT 2025/7

Cyclotomic products: Solution, step 1

Trade squares for ordered pairs. Each pair $i < j$ gives two ordered differences:

$$(\omega^i - \omega^j)(\omega^j - \omega^i) = -(\omega^i - \omega^j)^2.$$

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There are $\binom{5}{2} = 10$ pairs, so the signs multiply to $(-1)^{10} = 1$:

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Why: now the product splits into five pieces, one per j, and each piece runs over *all the other* roots — the shape the model product handles.

Cyclotomic products: Solution, step 2

Factor out the common power. $\omega^j - \omega^i = \omega^j(1 - \omega^{i-j})$. *Why:* we want every factor to look like $1 - (\text{root of unity})$.

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As i runs over $\{1, \ldots, 5\} \setminus \{j\}$, $i - j$ hits each nonzero residue mod 5 once, so the leftovers are exactly $1 - \omega, \ldots, 1 - \omega^4$:

$$\prod_{i \neq j} (\omega^j - \omega^i) = \omega^{4j} \prod_{k=1}^{4} (1 - \omega^k) = 5\,\omega^{4j} \quad \text{(model product, } n = 5\text{)}.$$

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Multiply over j, using $\omega^5 = 1$:

$$\prod_{j=1}^{5} 5\,\omega^{4j} = 5^5 \cdot \omega^{4(1+2+3+4+5)} = 5^5 \cdot (\omega^5)^{12} = \boxed{3125}.$$

Practice Exercises

One problem to try — the model product, used directly.

Practice: the model product

Exercise. Let $\omega = e^{2\pi i/7}$. Compute $\prod_{k=1}^{6}(1-\omega^k)$.

Hint: Evaluate $x^6 + x^5 + \cdots + x + 1 = \prod_{k=1}^{6}(x-\omega^k)$ at $x = 1$.

Answers

1. Exercise: $\boxed{7}$