Vieta on Cyclotomic Polynomials

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A product over roots of unity is a polynomial evaluated at a point — write the polynomial whose roots they are, then plug in.

Today's question: evaluate a product like $\prod_{k=1}^{6}(1-\omega^k)$, where $\omega = e^{2\pi i/7} = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The numbers $1, \omega, \omega^2, \ldots, \omega^6$ are the *seventh roots of unity*: the seven complex solutions of $z^7 = 1$, spaced evenly around the unit circle. The core move: the polynomial $x^6 + x^5 + \cdots + x + 1$ has exactly $\omega, \omega^2, \ldots, \omega^6$ as its roots, so writing that polynomial in factored form turns the whole product into a single evaluation at $x = 1$. Concretely, $\prod_{k=1}^{6}(1-\omega^k)$ is just the polynomial $\prod_{k=1}^{6}(x-\omega^k)$ evaluated at $x = 1$. We work one contest example fully, then you try the drill.

Technique: Vieta on cyclotomic polynomials

The n-th roots of unity are the roots of $x^n - 1$, and removing $x = 1$ leaves the factor $x^{n-1} + \cdots + x + 1$ whose roots are all the remaining n-th roots of unity. Every monic polynomial is the product of x minus its roots (the factored form behind Vieta's formulas), so plugging a number into the factored form of *these* polynomials evaluates the products $\prod(1-\omega^i)$ that appear everywhere. The model computation: for $\omega = e^{2\pi i/n}$ a primitive n-th root,

$$\prod_{i=1}^{n-1}(1-\omega^i) = n,$$

by evaluating $x^{n-1} + \cdots + x + 1 = \prod_{i=1}^{n-1}(x-\omega^i)$ at $x = 1$.

Problem 1 (BMT 2025/7)**.** Let $\omega = e^{2\pi i/5}$. Compute

$$\prod_{1\le i<j\le 5}(\omega^i - \omega^j)^2.$$

Solution. First trade the squares for ordered pairs. Each unordered pair $i < j$ contributes two ordered differences, $(\omega^i-\omega^j)(\omega^j-\omega^i) = -(\omega^i-\omega^j)^2$. There are $\binom{5}{2} = 10$ pairs, so the minus signs multiply to $(-1)^{10} = 1$, and

$$\prod_{i<j}(\omega^i-\omega^j)^2 = \prod_{j=1}^{5}\prod_{i\neq j}(\omega^j-\omega^i).$$

Fix j and factor ω^j out of each term: $\omega^j - \omega^i = \omega^j(1-\omega^{i-j})$. As i runs over $\{1,\ldots,5\}\setminus\{j\}$, the exponent $i-j$ runs through the nonzero residues mod 5, each exactly once, so

$$\prod_{i\neq j}(\omega^j-\omega^i) = \omega^{4j}\prod_{k=1}^{4}(1-\omega^k) = 5\,\omega^{4j},$$

using the model computation with $n = 5$. Multiplying over j:

$$\prod_{j=1}^{5} 5\,\omega^{4j} = 5^5 \cdot \omega^{4(1+2+3+4+5)} = 5^5 \cdot (\omega^5)^{12} = 3125,$$

since $\omega^5 = 1$. □

Factor the common power out first; the cyclotomic product does the rest.

Exercise 2. Let $\omega = e^{2\pi i/7}$. Compute $\prod_{k=1}^{6}(1 - \omega^k)$. Hint: 1

Answers: 7.

Hints

1. Evaluate $x^6 + x^5 + \cdots + x + 1 = \prod_{k=1}^{6}(x - \omega^k)$ at $x = 1$.