Counting Factorizations by Root Distribution

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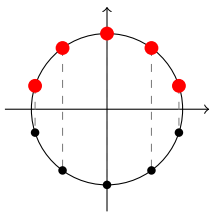
The key idea: don't work with coefficients — distribute the roots among the factors and count the distributions.

the ten roots of $x^{10} + 1$



one root from each conjugate pair

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Root Distribution

Each root, with its multiplicity, must go to one factor or another — count the distributions, not the coefficients.

The technique in two sentences

When to reach for it: a polynomial is forced to factor in a constrained way — $a^2 + b^2 = (a + bi)(a - bi)$, or $p(x) - p(2)$ with known roots — and you are asked to *count* the possibilities.

The move: a monic factor is decided by which roots it gets. So instead of solving for coefficients, hand each root (with its multiplicity) to one factor, respect the constraints, and count the ways.

Why it works: coefficients satisfy messy equations, but "which roots go where" is a clean counting problem.

Root distribution: the setup

A sum of two squares factors over **C**:

$$a^2 + b^2 = (a + bi)(a - bi).$$

If a, b are real, then $a - bi$ is the conjugate of $a + bi$ — so conjugation swaps the roots of $a + bi$ with the roots of $a - bi$.

The ten roots of $x^{10} + 1$ are nonreal and form five conjugate pairs (the dashed chords on the title slide). Since each pair is split between the two factors, choosing the five roots of $a + bi$ means choosing exactly one from each pair.

Root distribution: Example

Nikhil lists all real-coefficient polynomial pairs $(a(x), b(x))$ with $\deg a > \deg b$ and $a(x)^2 + b(x)^2 = x^{10} + 1$. Danielle adds all further pairs satisfying the same conditions with complex coefficients. Nikhil's list has N pairs; Danielle adds D. Compute $D - N$.

Strategy: the roots of $x^{10} + 1$ must be distributed between $a + bi$ and $a - bi$ — count the distributions. [1]

[1] BMT 2023/7

Root distribution: Try it

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Hint: $\deg a > \deg b$ forces $\deg a = 5$ and the leading coefficient of $a + bi$ to be ± 1. How many ways can five roots be chosen, with and without the one-from-each-pair rule? [2]

[2] BMT 2023/7

Root distribution: Solution, step 1

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Conversely every choice works: $a = \frac{(a+bi)+(a-bi)}{2}$, $b = \frac{(a+bi)-(a-bi)}{2i}$, and the leading terms cancel in b.

Root distribution: Solution, step 2

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$$\boxed{D - N = 440 - 64 = 376}.$$

Double-Root Shapes

When a polynomial takes the value $p(2)$ at exactly one other integer, $p(x) - p(2)$ has a repeated root — classify the double-root shapes.

Double-root shapes: Example

Find the number of cubic polynomials $p(x) = x^3 + ax^2 + bx + c$, where a, b, c are integers in $\{-20, -19, \ldots, 19, 20\}$, such that there is a unique integer $m \neq 2$ with $p(m) = p(2)$.

Strategy: $p(x) - p(2)$ has roots 2 and m, and its third root is an integer; uniqueness of m forces a double root — classify the two shapes. [3]

[3] 2023 AIME I/9

Double-root shapes: Try it

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Hint: The constant c is free (41 choices). Then $p(x) - p(2) = (x-2)^2(x-k)$ or $(x-2)(x-k)^2$ — expand each and bound b. [4]

[4] 2023 AIME I/9

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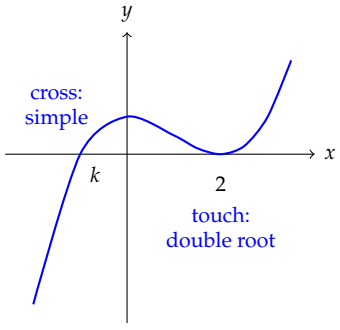
Why that matters: if the third root were a new number $m' \neq 2, m$, then $p(m') = p(2)$ too — and m would not be unique. So it repeats 2 or m:

$$p(x) - p(2) = (x-2)^2(x-k) \quad \text{or} \quad (x-2)(x-k)^2, \qquad k \neq 2.$$

Double-root shapes: Solution, step 2

Case $(x-2)^2(x-k)$**:**
$= x^3 - (k+4)x^2 + (4k+4)x - 4k$. Need $-20 \le 4k+4 \le 20$: $-6 \le k \le 4$, $k \neq 2$ — 10 values.



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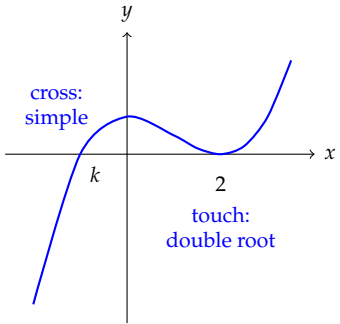
$= x^3 - (2k+2)x^2 + (k^2+4k)x - 2k^2$. Need $-20 \le k^2+4k \le 20$: $-6 \le k \le 2$, $k \neq 2$ — 8 values.

Why only b: in both cases a stays in range automatically.



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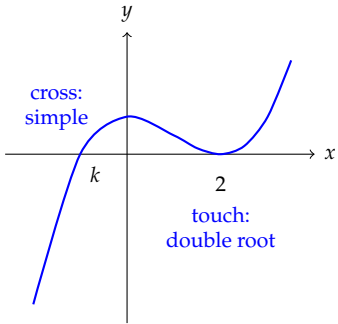
Total: $\boxed{41 \cdot (10+8) = 738}$.



Practice Exercises

One problem to try — root distribution on a smaller case.

Practice: root distribution

Exercise. Count the real-coefficient pairs $(a(x), b(x))$ with $\deg a > \deg b$ and $a(x)^2 + b(x)^2 = x^6 + 1$.

Hint: The six roots of $x^6 + 1$ form three conjugate pairs; choose one root from each, times 2 for the leading sign.

Answers

1. Exercise: $\boxed{16}$