Counting Factorizations by Root Distribution

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Don't work with coefficients — distribute the roots among the factors and count the distributions.

When a polynomial factors in a constrained way, the roots dictate the possibilities: each root (with its multiplicity) must go to one factor or another. Instead of working with coefficients, distribute the roots among the factors and count the distributions.

A sum of two squares factors over $\mathbf{C}$: $a^2 + b^2 = (a + bi)(a - bi)$. If a, b are real polynomials, then $a - bi$ is the conjugate of $a + bi$, so conjugation swaps the roots of $a + bi$ with the roots of $a - bi$. When the roots of $a^2 + b^2$ are all simple and nonreal, as below, this means the roots chosen for $a + bi$ must contain *exactly one root from each conjugate pair*.

Problem 1 (BMT 2023/7)**.** Nikhil lists all real-coefficient polynomial pairs $(a(x), b(x))$ with $\deg a > \deg b$ and $a(x)^2 + b(x)^2 = x^{10} + 1$. Danielle adds all further pairs satisfying the same conditions with complex coefficients. Nikhil's list has N pairs; Danielle adds D. Compute $D - N$.

Solution. Write $a^2 + b^2 = (a + bi)(a - bi) = x^{10} + 1$. Since $\deg a > \deg b$, the leading term of a^2 cannot cancel, so $2 \deg a = 10$ and $\deg a = 5$. Because $\deg b < 5$, the polynomials $a + bi$ and $a - bi$ both have degree 5 and the *same* leading coefficient λ (that of a); their product is monic, so $\lambda^2 = 1$ and $\lambda = \pm 1$. Hence $a + bi$ is ± 1 times a product $\prod(x - r)$ over exactly five of the ten roots of $x^{10} + 1$, and $a - bi$ is the same sign times the product over the other five. Conversely, each such choice gives a pair: $a = \frac{(a+bi)+(a-bi)}{2}$ and $b = \frac{(a+bi)-(a-bi)}{2i}$, with $\deg b < 5$ because the leading terms cancel.

Real case: the ten roots $e^{\pi i(2k+1)/10}$ are nonreal and form five conjugate pairs, and the five roots contain exactly one from each pair: 2^5 choices, times 2 for the sign. $N = 64$.

Complex case: any five of ten roots: $\binom{10}{5} = 252$ choices, times 2. Danielle's total is 504, so $D = 440$ and $D - N = 376$. □

The same root-distribution idea counts polynomials with prescribed values: if $p(m) = p(2)$ for a unique integer $m \neq 2$, then $p(x) - p(2)$ has roots 2 and m, forcing a repeated-root structure.

Problem 2 (2023 AIME I/9)**.** Find the number of cubic polynomials $p(x) = x^3 + ax^2 + bx + c$, where a, b, c are integers in $\{-20, -19, \ldots, 19, 20\}$, such that there is a unique integer $m \neq 2$ with $p(m) = p(2)$.

Solution. The constant c is free: 41 choices. Now $p(x) - p(2)$ is a cubic with roots 2 and m. By Vieta its third root is $-a - 2 - m$, an integer; if it were a new number, it would be a second integer $m' \neq 2$ with $p(m') = p(2)$. So uniqueness of m forces the third root to repeat 2 or m: $p(x) - p(2) = (x - 2)^2(x - k)$ or $(x - 2)(x - k)^2$ for some integer $k \neq 2$.

Case $(x-2)^2(x-k) = x^3-(k+4)x^2+(4k+4)x-4k$: need $-20 \le 4k+4 \le 20$, so $-6 \le k \le 4$, $k \neq 2$: 10 values.

Case $(x-2)(x-k)^2 = x^3-(2k+2)x^2+(k^2+4k)x-2k^2$: need $-20 \le k^2+4k \le 20$, so $-6 \le k \le 2$, $k \neq 2$: 8 values.

Total: $41 \cdot (10+8) = 738$. □

Exercise 3. Count the real-coefficient pairs $(a(x), b(x))$ with $\deg a > \deg b$ and $a(x)^2 + b(x)^2 = x^6 + 1$. Hint: 1

Answers: 16.

Hints

1. The six roots of x^6+1 form three conjugate pairs; choose one root from each, times 2 for the leading sign.