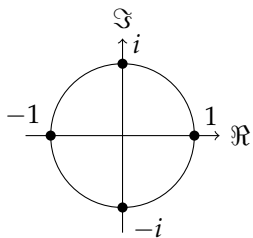
The Roots-of-Unity Filter

David Liu

The key idea: to isolate the terms with $m \mid k$ in a sum, average over m-th roots of unity — the filter turns "$m \mid k$" into algebra.



the 4th roots of unity

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The Filter

Average over m-th roots of unity to keep only the terms whose index is divisible by m.

The technique in two sentences

When to reach for it: a sum, probability, or count where only the indices k with $m \mid k$ (or $k \equiv r \pmod{m}$) matter — "heads a multiple of 4", "sum divisible by 3".

The move: plug every m-th root of unity into the generating function (binomial, product, ...) and average. The terms you want survive with weight 1; all others cancel to 0.

Why it works: for $\omega = e^{2\pi i/m}$, $1 + \omega^k + \omega^{2k} + \cdots + \omega^{(m-1)k}$ is m if $m \mid k$; otherwise it is a geometric sum $\frac{\omega^{km}-1}{\omega^k-1} = 0$.

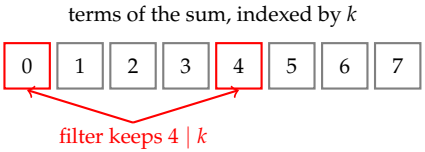
The roots-of-unity filter: the setup

To isolate the terms of a sum whose index satisfies $m \mid k - r$, average over m-th roots of unity, $\omega = e^{2\pi i/m}$:

$$\frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jr}\, \omega^{jk} = \begin{cases} 1 & m \mid k - r \\ 0 & \text{otherwise.} \end{cases}$$

Applied to a binomial expansion, the inner sum collapses back into a power.

Today's question: a biased coin is flipped 100 times — what is the probability that the number of heads is a multiple of 4?



The Coin Problem

Filter the binomial expansion, then read each surviving term as a power.

The coin problem: Example

A biased coin lands heads with probability $\frac{6+2\sqrt{3}}{12}$. After 100 flips, the probability that the number of heads is a multiple of 4 can be written as $\frac{1}{4} + \frac{1+a^b}{c \cdot d^e}$ with a, b, c, d, e positive integers. Find the minimal possible value of $a + b + c + d + e$.

Strategy: filter the binomial sum with 4th roots of unity, evaluate the four surviving powers, then match the required form. [1]

[1] BMT 2019/8

The coin problem: Try it

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Hint: With $p = \frac{3+\sqrt{3}}{6}$, $q = \frac{3-\sqrt{3}}{6}$, the probability is $\frac{1}{4}\sum_{j=0}^{3}(q + i^j p)^{100}$. Find $|q + ip|$ and $\arg(q + ip)$. [2]

[2] BMT 2019/8

The coin problem: Solution (filtering)

Let $p = \frac{3+\sqrt{3}}{6}$, $q = \frac{3-\sqrt{3}}{6}$. The desired probability is $\sum_{4|k} \binom{100}{k} p^k q^{100-k}$.

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Filter. The generating function is $(q+px)^{100} = \sum_k \binom{100}{k} p^k q^{100-k} x^k$. Plug in $x = 1, i, -1, -i$ and average:

$$P = \frac{1}{4}\sum_{j=0}^{3}(q+i^j p)^{100} = \frac{1}{4}\Big[1 + (p-q)^{100} + (q+ip)^{100} + (q-ip)^{100}\Big].$$

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Why: the filter has turned a sum of 26 terms into four powers — and powers of complex numbers are easy in polar form.

The coin problem: Solution (the four powers)

Polar form. $|q+ip|^2 = p^2+q^2 = \frac{2}{3}$ and $\frac{p}{q} = 2+\sqrt{3} = \tan 75^\circ$, so $q+ip = \sqrt{\frac{2}{3}}\, e^{5\pi i/12}$.

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Conjugate pair. $q-ip$ is the conjugate, so the two powers add to twice a real part. With $100 \cdot \frac{5\pi}{12} = \frac{125\pi}{3}$ and $\cos\frac{125\pi}{3} = \cos 300^\circ = \frac{1}{2}$:

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Real root. $(p-q)^{100} = \left(\frac{\sqrt{3}}{3}\right)^{100} = \left(\frac{1}{3}\right)^{50}$, so

$$P = \frac{1}{4}\left[1 + \left(\tfrac{1}{3}\right)^{50} + \left(\tfrac{2}{3}\right)^{50}\right] = \frac{1}{4} + \frac{1+2^{50}}{4\cdot 3^{50}}.$$

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For $k \geq 2$: $a^b + 1 = k(2^{50} + 1)$ with $a + b \leq 67$ forces $(a, b) = (8, 50)$ or $(32, 30)$. Then $a + b \geq 58$ leaves $c + d + e \leq 9$, so $c \cdot d^e \leq 4^4 = 256$ — but we need $c \cdot d^e > 7^8$. So $k = 1$.

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Minimal choice $(a,b,c,d,e) = (4,25,4,9,25)$:

$$\boxed{4+25+4+9+25 = 67}.$$

Practice Exercises

Two problems to try — the filter on a product and on a trig sum.

Practice: subset sums

Exercise 1. How many subsets of $\{1, 2, \ldots, 100\}$ have element-sum divisible by 3?

Hint: Filter $\prod_{k=1}^{100}(1 + z^k)$ at $1, \omega, \omega^2$. For $z = \omega$, group the factors in triples by k mod 3.

Practice: a trig sum

Exercise 2. Compute $\left\lfloor \sum_{k=0}^{10} \left(3 + 2\cos\frac{2k\pi}{11}\right)^{10} \right\rfloor \pmod{100}$.

Hint: Write $3 + 2\cos\frac{2k\pi}{11} = 3 + \zeta^k + \zeta^{-k}$ with $\zeta = e^{2\pi i/11}$. Expand $(3 + \zeta^k + \zeta^{-k})^{10}$ as $\sum \binom{10}{a,b,c} 3^a \zeta^{k(b-c)}$ over $a + b + c = 10$, sum over k, and keep the terms with $11 \mid (b - c)$. [3]

[3] PUMaC 2023 Algebra #A5

Answers

1. Exercise 1: $\boxed{\dfrac{2^{100}+2^{33}}{3}}$

2. Exercise 2: $\boxed{91}$