The Roots-of-Unity Filter

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To keep only the terms of a sum whose index is divisible by m, average the sum over the m-th roots of unity.

Today's question: a biased coin is flipped 100 times; what is the probability that the number of heads is a multiple of 4? That probability is a sum $\sum_k \binom{100}{k} p^k q^{100-k}$ over only those k divisible by 4, but the binomial expansion naturally sums over *all* k. The *roots-of-unity filter* fixes this: the m-th roots of unity are the m complex solutions of $z^m = 1$, and averaging ω^{jk} over them equals 1 when m divides k and 0 otherwise, plucking out exactly the terms you want. We work the coin problem fully, then you try two exercises: a subset count and a trig sum.

Technique: The roots-of-unity filter

To isolate the terms of a sum whose index is divisible by m, average over m-th roots of unity:

$$\frac{1}{m}\sum_{j=0}^{m-1} \omega^{-jr}\,\omega^{jk} = \begin{cases} 1 & m \mid k-r \\ 0 & \text{otherwise} \end{cases}, \qquad \omega = e^{2\pi i/m}.$$

(Why: if $m \mid k-r$, every term is 1; otherwise $z = \omega^{k-r} \neq 1$ and the geometric sum $1 + z + \cdots + z^{m-1} = \frac{z^m-1}{z-1}$ is 0 because $z^m = 1$.) Applied to a binomial expansion, the inner sum collapses back into a power: the filter turns "$m \mid k$" into algebra.

Problem 1 (BMT 2019/8)**.** A biased coin lands heads with probability $\frac{6+2\sqrt{3}}{12}$. After 100 flips, the probability that the number of heads is a multiple of 4 can be written as $\frac{1}{4} + \frac{1+a^b}{c\cdot d^e}$ with a, b, c, d, e positive integers. Find the minimal possible value of $a+b+c+d+e$.

Solution. Let $p = \frac{3+\sqrt{3}}{6}$, $q = \frac{3-\sqrt{3}}{6}$. The desired probability is $\sum_{4\mid k} \binom{100}{k} p^k q^{100-k}$. Filtering with 4th roots of unity,

$$P = \frac{1}{4}\sum_{j=0}^{3} (q + i^j p)^{100} = \frac{1}{4}\Big[1 + (p-q)^{100} + (q+ip)^{100} + (q-ip)^{100}\Big].$$

Now $|q+ip|^2 = p^2 + q^2 = \frac{2}{3}$, and recognizing $\frac{p}{q} = 2 + \sqrt{3} = \tan 75^\circ$ (the standard exact value) gives $q + ip = \sqrt{\frac{2}{3}}\, e^{5\pi i/12}$. Hence

$$(q+ip)^{100} + (q-ip)^{100} = 2\left(\tfrac{2}{3}\right)^{50} \cos\frac{125\pi}{3} = \left(\tfrac{2}{3}\right)^{50},$$

since $\cos\frac{125\pi}{3} = \cos 300^\circ = \frac{1}{2}$. With $(p-q)^{100} = (\frac{\sqrt{3}}{3})^{100} = (\frac{1}{3})^{50}$,

$$P = \frac{1}{4}\left[1 + \left(\tfrac{1}{3}\right)^{50} + \left(\tfrac{2}{3}\right)^{50}\right] = \frac{1}{4} + \frac{1+2^{50}}{4\cdot 3^{50}}.$$

The fraction is in lowest terms ($\gcd(1+2^{50}, 4\cdot 3^{50}) = 1$), so every representation scales numerator and denominator by the same $k \geq 1$. For $k \geq 2$: if $a+b \geq 68$ the total already exceeds 67; otherwise $a^b + 1 = k(2^{50}+1)$ with $a+b \leq 67$ forces $(a,b) = (8,50)$ or $(32,30)$, where $k = 2^{100} - 2^{50} + 1$ and $c \cdot d^e = 4k \cdot 3^{50} > 7^8$; but $a+b \geq 58$ leaves $c+d+e \leq 9$, and $c+d+e \leq 9$ gives $c \cdot d^e \leq 4^4 = 256$ (try the few cases), a contradiction; hence $k = 1$ is smallest. The exponent splittings of 2^{50} are $(2,50)$, $(4,25)$, $(32,10)$, $(1024,5)$ — $(4,25)$ wins with the smallest sum 29 — and for the denominator $(4,9,25)$ totals 38, beating $(4,3,50) \to 57$, $(12,3,49) \to 64$, and the rest. So the minimal choice is $(a,b,c,d,e) = (4,25,4,9,25)$, giving $4+25+4+9+25 = 67$. □

Exercise 2. How many subsets of $\{1,2,\ldots,100\}$ have element-sum divisible by 3? Hint: 1

Exercise 3 (PUMaC 2023 Algebra #A5)**.** Compute $\left\lfloor \sum_{k=0}^{10} \left(3 + 2\cos\frac{2k\pi}{11}\right)^{10} \right\rfloor \pmod{100}$. Hint: 2

Answers: $\dfrac{2^{100}+2^{33}}{3}$; 91.

Hints

1. Filter $\prod_{k=1}^{100}(1+z^k)$ at $1, \omega, \omega^2$. For $z = \omega$, group the factors in triples by k mod 3.
2. With $\zeta = e^{2\pi i/11}$, write the base as $3 + 2\cos\frac{2k\pi}{11} = 3 + \zeta^k + \zeta^{-k}$. Expand $(3+\zeta^k+\zeta^{-k})^{10}$ as $\sum \binom{10}{a,b,c} 3^a \zeta^{k(b-c)}$ over $a+b+c = 10$, sum over k, and keep the terms with $11 \mid (b-c)$.