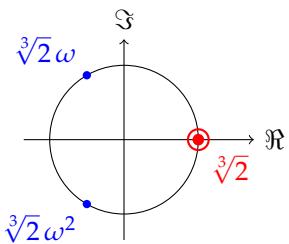
Reality Forcing

David Liu

The key idea: when a complex expression must be real, write it out and kill the imaginary part — the nonreal choices die.



the cube roots of 2 — only the real one survives

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Kill the Imaginary Part

A complex expression must be real — write it out, kill the imaginary part, keep the survivors.

The technique in two sentences

When to reach for it: the unknowns are complex, but the question asks only for the *real* values of some expression.

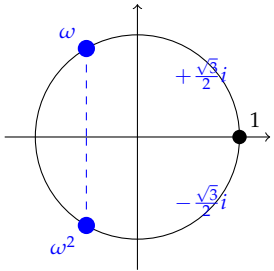
The move: solve as far as you can over $\mathbf{C}$ (cube roots come with factors $1, \omega, \omega^2$), write the expression as (real part) $+ i$ (imaginary part), and set the imaginary part to 0. That equation is usually tiny, so you can list its solutions.

Why it works: "real" is one extra equation. It throws out the nonreal choices, and only a few survive.

Reality forcing: the setup

Spot the perfect cube:

$$a^3 + b^3 + 3ab(a+b) = (a+b)^3.$$



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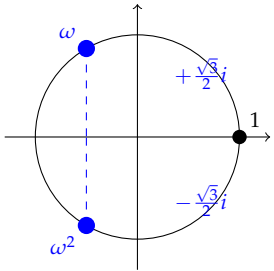
So $a^3 + b^3 = 2 - 3ab(a+b)$ becomes
$(a+b)^3 = 2$, and 2 has three cube roots:

$$a + b = \sqrt[3]{2}\,\omega^r, \qquad r \in \{0,1,2\},\ \omega = e^{2\pi i/3}.$$



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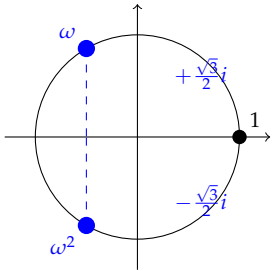
For a real answer, the imaginary part must vanish. ω and ω^2 have imaginary parts $\pm\frac{\sqrt{3}}{2}$, so a sum of such terms is real only when these cancel.



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Reality forcing: Example

Complex numbers a, b, c satisfy $a^3 + b^3 = 2 - 3ab(a + b)$, $b^3 + c^3 = 16 - 3bc(b + c)$, $c^3 + a^3 = 54 - 3ca(c + a)$. Compute the sum of all distinct possible real values of $a + b + c$.

Strategy: each equation is a perfect cube — take cube roots, then kill the imaginary part of $a + b + c$. [1]

[1] BMT 2025/5

Reality forcing: Try it

Complex numbers a, b, c satisfy $a^3 + b^3 = 2 - 3ab(a+b)$, $b^3 + c^3 = 16 - 3bc(b+c)$, $c^3 + a^3 = 54 - 3ca(c+a)$. Compute the sum of all distinct possible real values of $a + b + c$.

Hint: $(a+b)^3 = 2$, $(b+c)^3 = 16$, $(c+a)^3 = 54$. Write $a + b + c = \frac{\sqrt[3]{2}}{2}\left(\omega^r + 2\omega^s + 3\omega^t\right)$ — when is its imaginary part zero? [2]

[2] BMT 2025/5

Reality forcing: Solution, step 1

Take cube roots. $(a+b)^3 = 2$, $(b+c)^3 = 16 = 2^3 \cdot 2$, $(c+a)^3 = 54 = 3^3 \cdot 2$, so with $\omega = e^{2\pi i/3}$

$$a+b = \sqrt[3]{2}\,\omega^r, \quad b+c = 2\sqrt[3]{2}\,\omega^s, \quad c+a = 3\sqrt[3]{2}\,\omega^t, \qquad r,s,t \in \{0,1,2\}.$$

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Why we may pick freely: the three pair sums determine a, b, c, so all 27 choices of (r,s,t) really occur.

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Add and halve.

$$a+b+c = \tfrac{\sqrt[3]{2}}{2}\left(\omega^r + 2\omega^s + 3\omega^t\right).$$

Reality forcing: Solution, step 2

Kill the imaginary part. Let c_1, c_2, c_3 be 0, 1, or -1 according as r, s, t is 0, 1, or 2 (the imaginary part of ω^r is $\frac{\sqrt{3}}{2}c_1$, and so on). Reality forces

$$c_1 + 2c_2 + 3c_3 = 0.$$

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$$c_1 + 2c_2 + 3c_3 = 0.$$

Why this is tiny: $c_3 = 0$ forces $c_1 = c_2 = 0$; $c_3 = \pm 1$ forces $c_1 = c_2 = \mp 1$. Only three patterns survive: $(0, 0, 0)$, $(1, 1, -1)$, $(-1, -1, 1)$.

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Read off the values. $(0,0,0)$ gives $\frac{\sqrt[3]{2}}{2} \cdot 6 = 3\sqrt[3]{2}$; the other two give $\frac{\sqrt[3]{2}}{2}(3\omega + 3\omega^2) = -\frac{3\sqrt[3]{2}}{2}$.

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$$\boxed{3\sqrt[3]{2} - \frac{3\sqrt[3]{2}}{2} = \frac{3\sqrt[3]{2}}{2}}.$$

A Second Flavor: Vieta Elimination

When the equations are symmetric in the unknowns, rewrite them in $s = x + y$ and $p = xy$, solve for s, then keep the real values.

Vieta elimination: Example

Complex numbers x, y satisfy $x^2 + y^2 = 7$ and $x^3 + y^3 = 10$. Find the largest real value of $x + y$.

Strategy: set $s = x + y$, $p = xy$ — eliminate p, solve for s, keep the real candidates. [3]

[3] 1983 AIME/5

Vieta elimination: Try it

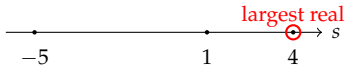
Complex numbers x, y satisfy $x^2 + y^2 = 7$ and $x^3 + y^3 = 10$. Find the largest real value of $x + y$.

Hint: $s^2 - 2p = 7$ and $s^3 - 3ps = 10$. Eliminate p and factor the resulting cubic in s.
[4]

[4] 1983 AIME/5

Vieta elimination: Solution

Put $s = x + y$, $p = xy$. Then $s^2 - 2p = 7$ and $s^3 - 3ps = 10$. *Why:* both equations are symmetric, and symmetric expressions are polynomials in s and p.



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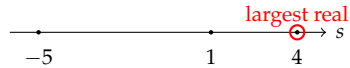

Eliminating p: $p = \frac{s^2-7}{2}$, so $s^3 - \frac{3s(s^2-7)}{2} = 10$, i.e.

$$s^3 - 21s + 20 = 0, \quad \text{i.e. } (s-1)(s-4)(s+5) = 0.$$

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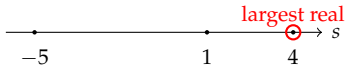

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The real candidates are $1, 4, -5$ (each realizable: x, y are the roots of $t^2 - st + p$); the largest is

$$\boxed{4}.$$

Practice Exercises

One problem to try — the perfect cube and its three cube roots.

Practice: one perfect cube

Exercise. Complex numbers a, b satisfy $a^3 + b^3 = 2 - 3ab(a + b)$. Find all real values of $a + b$.

Hint: $(a + b)^3 = 2$, so $a + b = \sqrt[3]{2}\,\omega^r$ with $r \in \{0, 1, 2\}$. Real only for $r = 0$.

Answers

1. Exercise: $\boxed{\sqrt[3]{2}}$