Reality Forcing

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When a complex expression must be real, write it out and kill the imaginary part. The nonreal choices die; only a few survive.

Problem 1 (BMT 2025/5)**.** Complex numbers a, b, c satisfy $a^3 + b^3 = 2 - 3ab(a+b)$, $b^3 + c^3 = 16 - 3bc(b+c)$, $c^3 + a^3 = 54 - 3ca(c+a)$. Compute the sum of all distinct possible real values of $a + b + c$.

Solution. Since $a^3 + b^3 + 3ab(a+b) = (a+b)^3$, each equation is a perfect cube: $(a+b)^3 = 2$, $(b+c)^3 = 16 = 2^3 \cdot 2$, $(c+a)^3 = 54 = 3^3 \cdot 2$. Each cube has three cube roots, so with $\omega = e^{2\pi i/3}$, $a + b = \sqrt[3]{2}\,\omega^r$, $b + c = 2\sqrt[3]{2}\,\omega^s$, $c + a = 3\sqrt[3]{2}\,\omega^t$ for some $r, s, t \in \{0, 1, 2\}$. Every choice of (r, s, t) actually occurs, since the three pair sums determine a, b, c (for instance $a = \frac{(a+b)+(c+a)-(b+c)}{2}$). Adding the three sums and halving,

$$a + b + c = \tfrac{\sqrt[3]{2}}{2}\left(\omega^r + 2\omega^s + 3\omega^t\right), \quad r, s, t \in \{0, 1, 2\}.$$

For this to be real, the imaginary part must vanish. Since $\operatorname{Im} 1 = 0$ and $\operatorname{Im} \omega = -\operatorname{Im} \omega^2 = \frac{\sqrt{3}}{2}$, let c_1, c_2, c_3 be 0, 1, or -1 according as r, s, t (respectively) is 0, 1, or 2. Then

$$\operatorname{Im}(a + b + c) = \tfrac{\sqrt[3]{2}}{2} \cdot \tfrac{\sqrt{3}}{2}\,(c_1 + 2c_2 + 3c_3),$$

so reality forces $c_1 + 2c_2 + 3c_3 = 0$. If $c_3 = 0$ then $c_1 = -2c_2$ forces $c_1 = c_2 = 0$; if $c_3 = \pm 1$ then $c_1 + 2c_2 = \mp 3$ forces $c_1 = c_2 = \mp 1$. So only three patterns survive: $(0, 0, 0)$, $(1, 1, -1)$, $(-1, -1, 1)$, i.e. $(r, s, t) = (0, 0, 0)$, $(1, 1, 2)$, $(2, 2, 1)$.

The first gives $\frac{\sqrt[3]{2}}{2}(1 + 2 + 3) = 3\sqrt[3]{2}$. The second gives $\frac{\sqrt[3]{2}}{2}(3\omega + 3\omega^2) = -\frac{3\sqrt[3]{2}}{2}$ (as $\omega + \omega^2 = -1$), and the third gives the same value. The distinct real values are $3\sqrt[3]{2}$ and $-\frac{3\sqrt[3]{2}}{2}$, whose sum is $\frac{3\sqrt[3]{2}}{2}$. □

Problem 2 (1983 AIME/5)**.** Complex numbers x, y satisfy $x^2 + y^2 = 7$ and $x^3 + y^3 = 10$. Find the largest real value of $x + y$.

Solution. Put $s = x + y$, $p = xy$. Then $s^2 - 2p = 7$ and $s^3 - 3ps = 10$. Eliminating p gives $s^3 - 21s + 20 = 0$, i.e. $(s-1)(s-4)(s+5) = 0$. The real candidates are $1, 4, -5$; the largest is 4. □

Exercise 3. Complex numbers a, b satisfy $a^3 + b^3 = 2 - 3ab(a+b)$. Find all real values of $a + b$. Hint: 1

Answers: $\sqrt[3]{2}$.

Hints

1. $(a+b)^3 = 2$, so $a+b = \sqrt[3]{2}\,\omega^r$ with $r \in \{0,1,2\}$. Real only for $r = 0$.