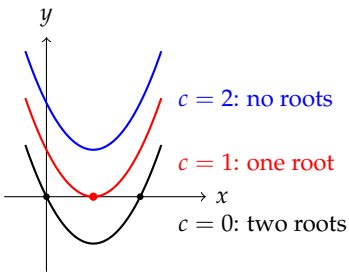
The Discriminant Method

David Liu



The key idea: "a real solution exists" *is* the condition discriminant ≥ 0.

$y = x^2 - 2x + c$ has real roots
$\iff \Delta = 4 - 4c \geq 0.$

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Symmetric Bounds

Isolate one variable; the reality of the others is a discriminant condition.

Symmetric bounds: the move

Isolate one variable; reality of the others is a discriminant condition.

When: symmetric equations in real variables, and you want the range of one of them.

The move: write the sum s and product p of the *other two* in terms of your target. Real numbers with sum s and product p are the roots of $z^2 - sz + p = 0$, so they exist $\iff s^2 \geq 4p$. That is an inequality in the target alone.

Symmetric bounds: Example

For real numbers B, M, T with $B^2 + M^2 + T^2 = 2022$ and $B + M + T = 72$, compute the sum of the minimum and maximum possible values of T.

Strategy: write $s = B + M$ and BM in terms of T; reality of B, M is $s^2 \geq 4BM$. [1]

[1] BMT 2022/5

Symmetric bounds: Try it

For real numbers B, M, T with $B^2 + M^2 + T^2 = 2022$ and $B + M + T = 72$, compute the sum of the minimum and maximum possible values of T.

Hint: $s = 72 - T$; express BM through $s^2 = B^2 + M^2 + 2BM$. You should get a quadratic inequality for T. [2]

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The extremes are the endpoints: $\boxed{\min T + \max T = 10 + 38 = 48}$.

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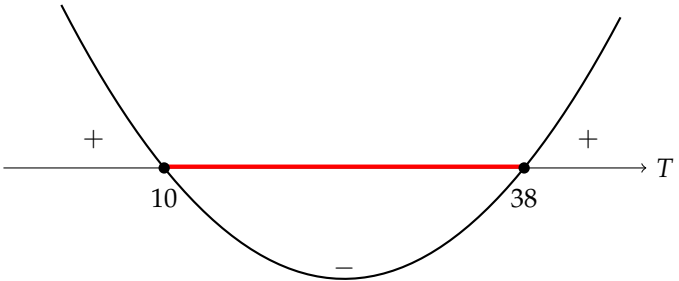

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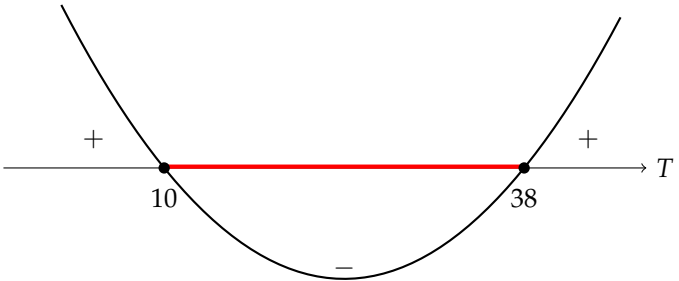

The quadratic is ≤ 0 exactly on $[10, 38]$ — so $\min T + \max T = 10 + 38 = \boxed{48}$.

Optimizing Ratios

Set $t = f/g$, substitute, and demand the resulting equation have a real solution.

Optimizing ratios: the move

To optimize a ratio $t = f/g$ under a constraint: set $f = tg$, substitute, demand a real solution.

When: maximize or minimize a ratio such as $\frac{x}{y}$ subject to a quadratic constraint.

Why it works: a value t is possible exactly when the substituted equation has a real solution. The substituted equation is usually quadratic in the other variable. "A real solution exists" becomes discriminant ≥ 0 — an inequality for t alone, whose endpoints are the extreme values.

Optimizing ratios: Example

Find the minimum value of $\frac{x}{y}$, where x and y are real numbers satisfying $x^2 + y^2 - 8x - 6y + 20 = 0$.

Strategy: set $t = x/y$, substitute, and demand the quadratic in y have a real solution.

Optimizing ratios: Try it

Find the minimum value of $\frac{x}{y}$, where x and y are real numbers satisfying $x^2 + y^2 - 8x - 6y + 20 = 0$.

Hint: $x = ty$ gives $(t^2 + 1)y^2 - (8t + 6)y + 20 = 0$. When does a quadratic have a real root? (Check $y = 0$ can't happen first.)

Optimizing ratios: Solution

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The smallest possible ratio is the left endpoint: $\boxed{\min \dfrac{x}{y} = \dfrac{1}{2}}$.

Optimizing ratios: the geometry

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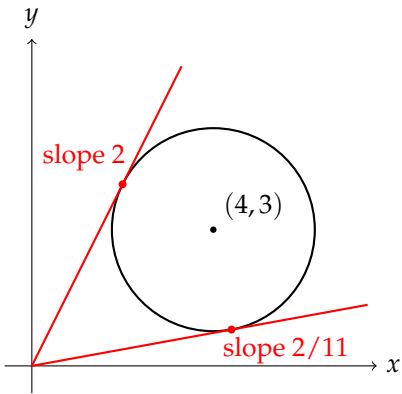

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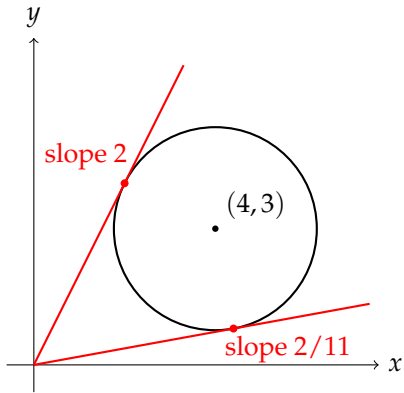

So $\frac{y}{x} \in [\frac{2}{11}, 2]$, i.e. $\frac{x}{y} \in [\frac{1}{2}, \frac{11}{2}]$ — $\Delta \geq 0$ *is* the tangency condition.

Practice Exercises

Two problems to try — the same moves you just saw.

Practice: symmetric bounds

Exercise 1. Real numbers a, b, c satisfy $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Find the maximum possible value of c.

Hint: Write $s = a + b = -c$ and ab in terms of c; reality of a, b is $s^2 \geq 4ab$. Check the bound is attained (take $a = b$).

Practice: reality of a quadratic

Exercise 2. Consider the parabola $(y + x) = (y - x)^2 + 3(y - x) + 3$. Find the minimum possible value of y.

Hint: Set $u = y - x$ to get a quadratic equation in u; a real point exists exactly when its discriminant is nonnegative.

Answers

1. Exercise 1: $\boxed{\sqrt{2/3}}$
2. Exercise 2: $\boxed{-1/2}$