The Discriminant Method

David Liu

Turn "a real solution exists" into discriminant ≥ 0.

A single weapon for many optimization problems: when the value you want is the one that still lets some quadratic have a real root, the discriminant condition gives its exact range. Two faces of it:

Face 1: symmetric bounds. Given symmetric equations in real variables, isolate one: write the sum s and product p of the *other two* in terms of your target. Real numbers with sum s and product p are the roots of $z^2 - sz + p = 0$, so they exist exactly when $s^2 \geq 4p$.

Face 2: ratios. To optimize a ratio $t = f/g$ subject to a constraint, set $f = tg$, substitute, and demand the resulting equation have a real solution.

Problem 1 (BMT 2022/5)**.** For real numbers B, M, T with $B^2 + M^2 + T^2 = 2022$ and $B + M + T = 72$, compute the sum of the minimum and maximum possible values of T.

Solution. Put $s = B + M = 72 - T$. Then $BM = \frac{(72-T)^2 - (2022-T^2)}{2} = T^2 - 72T + 1581$. Real B, M exist iff $(B - M)^2 \geq 0$, i.e. $(72 - T)^2 \geq 4(T^2 - 72T + 1581)$, so $T^2 - 48T + 380 \leq 0$. Roots $10, 38$; hence $\min T + \max T = 48$. □

Problem 2. Find the minimum value of $\frac{x}{y}$, where x and y are real numbers satisfying $x^2 + y^2 - 8x - 6y + 20 = 0$.

Solution. First, $y \neq 0$: $y = 0$ would give $x^2 - 8x + 20 = 0$, whose discriminant $64 - 80$ is negative, so no real x. Hence $t = x/y$ is defined. Set $x = ty$; substituting,

$$(t^2 + 1)y^2 - (8t + 6)y + 20 = 0.$$

Real y exists iff the discriminant is nonnegative:

$$(8t + 6)^2 - 80(t^2 + 1) \geq 0 \iff 4t^2 - 24t + 11 \leq 0 \iff \tfrac{1}{2} \leq t \leq \tfrac{11}{2}.$$

So $\min \frac{x}{y} = \frac{1}{2}$. (Geometrically: $1/t$ is a slope, and the extremal slopes are the two tangents from the origin to the circle $(x - 4)^2 + (y - 3)^2 = 5$ — the discriminant condition *is* the tangency condition.) □

Exercise 3. Real numbers a, b, c satisfy $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Find the maximum possible value of c. Hint: 1

Exercise 4 (HMMT 2014 Theme #8)**.** Consider the parabola $(y + x) = (y - x)^2 + 3(y - x) + 3$. Find the minimum possible value of y. Hint: 2

Answers: $\sqrt{\frac{2}{3}}$; $-\frac{1}{2}$.

Hints

1. Write $s = a + b = -c$ and ab in terms of c; reality of a, b is $s^2 \geq 4ab$. Check the bound is attained (take $a = b$).
2. Set $u = y - x$ to get a quadratic equation in u; a real point exists exactly when its discriminant is nonnegative.