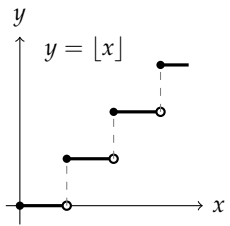
Floors: Sandwich, Then Verify

David Liu



The key idea: strip integer shifts, sandwich into a short interval, check candidates from the extreme end.

$\lfloor x \rfloor$ is constant on each $[n, n+1)$ — trap x, then check.

The Three-Step Recipe

Strip, sandwich, verify — and never forget the negative case.

The recipe: strip, sandwich, verify

Strip integer shifts, sandwich into a short interval, check candidates from the extreme end.

When: "find all / the greatest / the least x" with $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$ in the equation. Floors are hard to solve directly, but easy to bound: trap x first, then test the few candidates.

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2. **Sandwich:** $a - 1 < \lfloor a \rfloor \leq a$ and $a \leq \lceil a \rceil < a + 1$ trap x in a short interval.

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Two habits: bracket with nearby squares ($44^2 < 2022 < 45^2$ pins down $\lceil x \rceil$ instantly), and never forget the negative case.

Worked Example

Sandwich with nearby squares, then check the boundary.

Floors: Example

Find all real numbers x with $\lfloor x\lceil x\rceil\rfloor = 2022$, in interval notation.

Strategy: bracket with $44^2 < 2022 < 45^2$ to pin down $\lceil x\rceil$; handle $x \geq 0$ and $x < 0$ separately. [1]

[1] BMT 2022/4

Floors: Try it

Find all real numbers x with $\lfloor x\lceil x\rceil\rfloor = 2022$, in interval notation.

Hint: $\lfloor x\lceil x\rceil\rfloor = 2022$ means $2022 \leq x\lceil x\rceil < 2023$. If $x \geq 0$, why must $44 < x < 45$? [2]

[2] BMT 2022/4

Floors: Solution — the nonnegative case

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Why squares? For $x \ge 0$, $x\lceil x\rceil$ is roughly x^2, so compare 2022 with 44^2 and 45^2.

If $x \ge 0$: $x > 44$, else $x\lceil x\rceil \le 44 \cdot 44 = 1936 < 2022$.

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Hence $44 < x < 45$ and $\lceil x\rceil = 45$ — the ceiling is now a number, so just divide:

$$2022 \le 45x < 2023 \iff x \in \left[\tfrac{2022}{45}, \tfrac{2023}{45}\right) = \left[\tfrac{674}{15}, \tfrac{2023}{45}\right).$$

Floors: Solution — the negative case

Never skip this case. If $x < 0$: put $m = \lceil x \rceil \leq 0$. ($m = 0$ would give $\lfloor 0 \rfloor = 0 \neq 2022$, so $m < 0$.) *Plan:* squeeze m from both sides.

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$\lfloor mx \rfloor = 2022$ gives $2022 \leq mx < 2023$; dividing by $m < 0$ flips it. Also sandwich x by its ceiling:

$$\tfrac{2022}{m} \geq x > \tfrac{2023}{m}, \qquad m - 1 < x \leq m.$$

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Upper bound on mx: multiply $m - 1 < x$ by $m < 0$: $mx < m(m-1) = m^2 - m$. So $m^2 - m > 2022$; at $m = -44$ it is 1980, and it grows as m drops, so $m \leq -45$.

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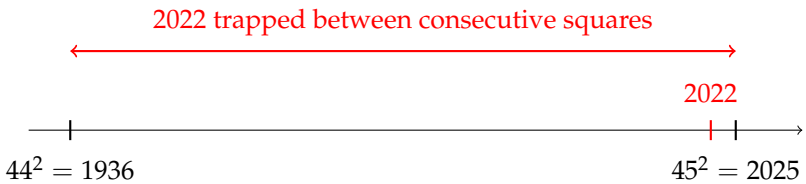
No negative solutions. The solution set is $\boxed{\left[\frac{674}{15}, \frac{2023}{45}\right)}$.

Floors: bracketing with nearby squares

The whole nonnegative case turned on one bracket:



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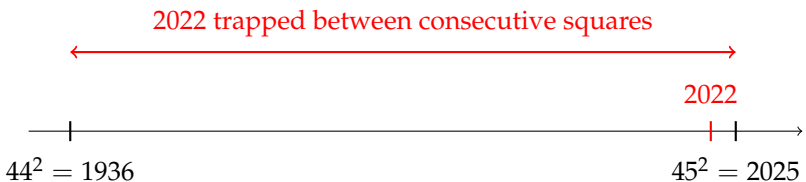

So $x > 44$ (else $x\lceil x \rceil \leq 1936$) and $x < 45$ (else $x\lceil x \rceil \geq 2025$) — $\lceil x \rceil = 45$ instantly, then one division.

Practice Exercises

Two problems to try — the same moves you just saw.

Practice: stripping shifts

Exercise 1. Find the greatest integer x such that $\left\lfloor \frac{x}{20} + 20 \right\rfloor = \left\lceil \frac{x}{24} + 24 \right\rceil$.

Hint: Strip the integer shifts to get $\lfloor x/20 \rfloor - \lceil x/24 \rceil = 4$. Sandwich to trap x below 720, then check downward from the top.

Practice: fixed exponents

Exercise 2. For how many positive integers $n \leq 1000$ does the equation $x^{\lfloor x \rfloor} = n$ have a positive real solution x?

Hint: On $[k, k+1)$ the exponent is fixed at k, so n must lie in $[k^k, (k+1)^k)$. Count the integers $n \leq 1000$ in each such interval; don't forget $0 < x < 1$, where $x^0 = 1$ gives only $n = 1$, already counted for $k = 1$.

Answers

1. Exercise 1: $\boxed{699}$
2. Exercise 2: $\boxed{412}$