Floors: Sandwich, Then Verify

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Strip integer shifts, sandwich x into a short interval, then check candidates from the extreme end.

$\lfloor \cdot \rfloor$ rounds down to the nearest integer; $\lceil \cdot \rceil$ rounds up.

For "greatest/least x satisfying a floor equation": (1) strip integer shifts, since $\lfloor u+n \rfloor = \lfloor u \rfloor + n$ for integer n; (2) sandwich with $a-1 < \lfloor a \rfloor \le a$ and $a \le \lceil a \rceil < a+1$ to trap x in a short interval; (3) walk in from the extreme end and check candidates directly — the boundary case often fails, and checking it is part of the solution, not an afterthought.

Two more habits: bracket with nearby squares ($44^2 < 2022 < 45^2$ pins down $\lceil x \rceil$ instantly), and never forget the negative case — it usually dies to a one-line contradiction, but you must write that line.

Problem 1 (BMT 2022/4)**.** Find all real numbers x with $\lfloor x\lceil x \rceil \rfloor = 2022$, in interval notation.

Solution. $\lfloor x\lceil x \rceil \rfloor = 2022$ gives $x\lceil x \rceil \ge 2022 > 0$.

If $x \ge 0$: then $x > 44$ (else $x\lceil x \rceil \le 44 \cdot 44 < 2022$), so $\lceil x \rceil \ge 45$; but $x \ge 45$ would give $x\lceil x \rceil \ge 45 \cdot 45 = 2025$, too big. Hence $44 < x < 45$ and $\lceil x \rceil = 45$. The equation becomes $2022 \le 45x < 2023$, i.e. $x \in \left[\frac{2022}{45}, \frac{2023}{45}\right) = \left[\frac{674}{15}, \frac{2023}{45}\right)$.

If $x < 0$: write $m = \lceil x \rceil \le 0$. ($m = 0$ gives $\lfloor 0 \rfloor = 0 \ne 2022$, so $m < 0$.) From $\lfloor mx \rfloor = 2022$ we get $2022 \le mx < 2023$, and $m < 0$ flips this to $\frac{2022}{m} \ge x > \frac{2023}{m}$. Also $m - 1 < x \le m$. Multiplying by $m < 0$ flips the inequality: $mx < m(m-1) = m^2 - m$. Since $mx \ge 2022$, we get $m^2 - m > 2022$, forcing $m \le -45$. But then $\frac{2023}{m} < x \le m$ would require $\frac{2023}{m} < m$, i.e. $2023 > m^2$ — impossible since $m^2 \ge 2025$. No negative solutions.

The solution set is $\left[\frac{674}{15}, \frac{2023}{45}\right)$. □

Exercise 2 (BMT 2024/4)**.** Find the greatest integer x such that $\left\lfloor \frac{x}{20} + 20 \right\rfloor = \left\lceil \frac{x}{24} + 24 \right\rceil$. Hint: 1

Exercise 3 (HMMT 2020 Team #1)**.** For how many positive integers $n \le 1000$ does the equation $x^{\lfloor x \rfloor} = n$ have a positive real solution x? Hint: 2

Answers: 699; 412.

Hints

1. Strip the integer shifts to get $\lfloor x/20 \rfloor - \lceil x/24 \rceil = 4$. Sandwich to trap x below 720, then check downward from the top: 699 works and nothing above it does.

2. On $[k, k+1)$ the exponent is fixed at k, so n must lie in $[k^k, (k+1)^k)$. Count the integers $n \leq 1000$ in each such interval; don't forget $0 < x < 1$, where $x^0 = 1$ gives only $n = 1$, already counted for $k = 1$.