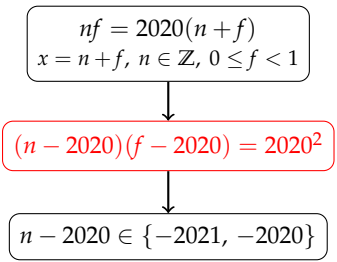
SFFT: Factoring Floor Equations

David Liu



The key idea: split $x = n + f$, complete to $(n - c)(f - c) = c^2$; the bound $0 \le f < 1$ leaves finitely many cases.

Simon says: factor it, then let the fractional part do the bounding.

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The Trick

An equation mixing $\lfloor x \rfloor$ and $\{x\}$ with a product term is begging to be completed into $(n-c)(f-c) = \text{const}$.

SFFT: the move

Split $x = n + f$; complete the product into $(n - c)(f - c) = \text{const}$.

When: an equation mixing $\lfloor x \rfloor$ and $\{x\}$ with a product term.

The move: write $x = n + f$ with $n \in \mathbb{Z}$ and $0 \leq f < 1$. Then $nf = c(n + f)$ becomes $nf - cn - cf = 0$; add c^2 to both sides and factor:

$$(n - c)(f - c) = c^2.$$

SFFT: the move

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Why it helps: f lives in a short interval, so the factor $f - c$ does too; then $n - c = c^2/(f - c)$ is trapped in a short interval, leaving only finitely many integers n — often just one or two.

Worked Example

The bound $0 \leq f < 1$ *turns an infinite equation into two cases.*

SFFT: Example

Compute the sum of the real solutions to $\lfloor x \rfloor \{x\} = 2020x$, where $\{x\} = x - \lfloor x \rfloor$.

Strategy: split $x = n + f$, complete into $(n - 2020)(f - 2020) = 2020^2$, and bound $f - 2020$. [1]

[1] BMT 2021/5

SFFT: Try it

Compute the sum of the real solutions to $\lfloor x \rfloor \{x\} = 2020x$, where $\{x\} = x - \lfloor x \rfloor$.

Hint: $x = n + f$ gives $(n - 2020)(f - 2020) = 2020^2$. What interval does $f - 2020$ live in? What does that force $n - 2020$ to be? [2]

[2] BMT 2021/5

SFFT: Solution — completing the product

Why split? $\lfloor x \rfloor$ and $\{x\}$ are the pieces of $x = n + f$, $n \in \mathbb{Z}$, $0 \leq f < 1$:

$$nf = 2020(n + f).$$

SFFT: Solution — completing the product

Why split? $\lfloor x \rfloor$ and $\{x\}$ are the pieces of $x = n + f$, $n \in \mathbb{Z}$, $0 \leq f < 1$:

$$nf = 2020(n + f).$$

Why factor? A product equal to a constant lets one factor control the other. Add 2020^2 (Simon's Favorite Factoring Trick):

$$nf - 2020n - 2020f + 2020^2 = 2020^2 \iff (n - 2020)(f - 2020) = 2020^2.$$

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Now $0 \le f < 1$ bounds a factor: $-2020 \le f - 2020 < -2019$. Dividing 2020^2 by it,

$$n - 2020 = \frac{2020^2}{f - 2020} \in \left(-2021 - \tfrac{1}{2019},\ -2020\right].$$

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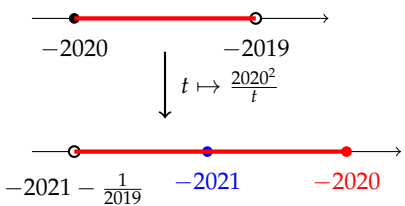
The integers in that half-open interval: -2021 and -2020. So $n = -1$ or $n = 0$.

SFFT: Solution — the two cases

$f - 2020 \in [-2020, -2019)$



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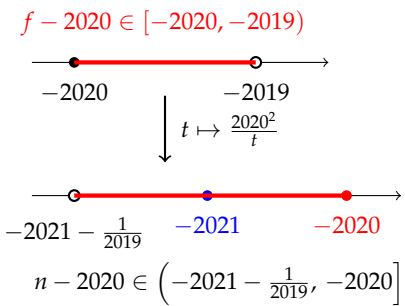
SFFT: Solution — the two cases



If $n = 0$: $(-2020)(f - 2020) = 2020^2$ gives $f = 0$, so $x = 0$.

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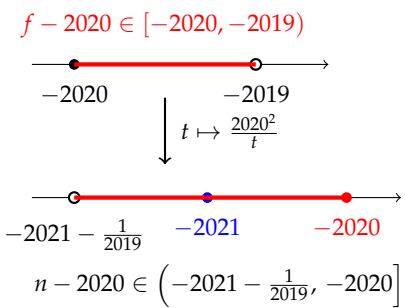

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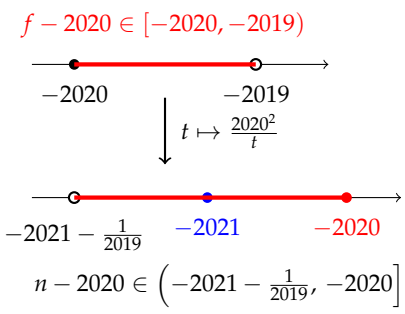

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Always check: both have $f \in [0, 1)$ and satisfy the original equation. Their sum is $\boxed{-\frac{1}{2021}}$.

Practice Exercises

Two problems to try — the same moves you just saw.

Practice: SFFT again

Exercise 1. Compute the sum of all real solutions to $\lfloor x \rfloor \{x\} = 2x$.

Hint: Write $x = n + f$; SFFT gives $(n-2)(f-2) = 4$. The bound $0 \le f < 1$ forces $n - 2 \in \{-3, -2\}$; solve for f in each case and check it lies in $[0, 1)$.

Practice: three-variable SFFT

Exercise 2. Suppose $a, b, c > 0$ are integers such that $abc - bc - ac - ab + a + b + c = 2013$. Find the number of ordered triples (a, b, c).

Hint: Add and subtract a constant to factor the left side into a product of three terms — expand $(a-1)(b-1)(c-1)$ and compare.

Answers

1. Exercise 1: $\boxed{-\frac{1}{3}}$
2. Exercise 2: $\boxed{18}$