SFFT: Factoring Floor Equations

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Split $x = n + f$, complete the product into $(n - c)(f - c) = \text{const}$, and let $0 \le f < 1$ do the bounding.

Here $\{x\} = x - \lfloor x \rfloor$ is the fractional part, $0 \le \{x\} < 1$.

Write $x = n + f$ with $n \in \mathbb{Z}$ and $0 \le f < 1$. An equation like $nf = c(n + f)$ rearranges into

$$(n - c)(f - c) = c^2,$$

Simon's Favorite Factoring Trick. (Expand to check: $(n - c)(f - c) = nf - cn - cf + c^2$.) Then $0 \le f < 1$ bounds the factor $f - c$, leaving only finitely many integer possibilities for n — often just one or two. Cue: an equation mixing $\lfloor x \rfloor$ and $\{x\}$ with a product term.

Problem 1 (BMT 2021/5)**.** Compute the sum of the real solutions to $\lfloor x \rfloor \{x\} = 2020x$, where $\{x\} = x - \lfloor x \rfloor$.

Solution. Write $x = n + f$ with $n \in \mathbb{Z}$ and $0 \le f < 1$. Then

$$nf = 2020(n + f) \iff nf - 2020n - 2020f = 0 \iff (n - 2020)(f - 2020) = 2020^2.$$

Since $0 \le f < 1$, we have $-2020 \le f - 2020 < -2019$, hence

$$n - 2020 = \frac{2020^2}{f - 2020} \in \left(\frac{2020^2}{-2019}, \frac{2020^2}{-2020} \right] = \left(-2021 - \tfrac{1}{2019}, -2020 \right].$$

The integers in this half-open interval are -2021 and -2020, so $n = -1$ or $n = 0$.

If $n = 0$: $(-2020)(f - 2020) = 2020^2$ gives $f = 0$, so $x = 0$. If $n = -1$: $(-2021)(f - 2020) = 2020^2$ gives $f = 2020 - \frac{2020^2}{2021} = \frac{2020}{2021}$, so $x = -1 + \frac{2020}{2021} = -\frac{1}{2021}$.

Both satisfy the original equation, and their sum is $-\frac{1}{2021}$. □

Exercise 2. Compute the sum of all real solutions to $\lfloor x \rfloor \{x\} = 2x$. Hint: 1

Exercise 3. Suppose $a, b, c > 0$ are integers such that $abc - bc - ac - ab + a + b + c = 2013$. Find the number of ordered triples (a, b, c). Hint: 2

Answers: $-\frac{1}{3}$; 18.

Hints

1. Write $x = n + f$; SFFT gives $(n-2)(f-2) = 4$. The bound $0 \leq f < 1$ forces $n - 2 \in \{-3, -2\}$; solve for f in each case and check it lies in $[0, 1)$.

2. Add and subtract a constant to factor the left side into a product of three terms — expand $(a-1)(b-1)(c-1)$ and compare.