AM-GM Under a Symmetric Constraint

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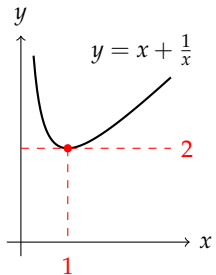

The key idea: bound the product by AM-GM, then read the answer off a monotone function.

$x + \frac{1}{x} \geq 2$: AM-GM turns a product bound into a clean minimum.

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The Move

A symmetric constraint plus a symmetric objective, where "by symmetry" feels unearned.

AM-GM: the move

Bound the product by AM-GM, then read the answer off a monotone function.

When: a symmetric constraint mixing $S = x^2 + y^2 + z^2$ and xyz, and you are asked for the extreme value of S. "By symmetry, $x = y = z$" gives the answer, but not a proof.

The move: AM-GM on x^2, y^2, z^2 trades the product for S:

$$|xyz| \leq \left(\frac{S}{3}\right)^{3/2}.$$

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Now the constraint involves S alone, as an inequality $C \leq g(S)$ with g increasing. Solve $g(S) = C$; that value is the bound — the rigorous version of "by symmetry."

Worked Example

Turn the constraint into a monotone inequality in S alone.

AM-GM: Example

Let x, y, z be real numbers with $7(x^2 + y^2 + z^2) + 9xyz = 12$. The minimum of $x^2 + y^2 + z^2$ equals a/b in lowest terms. Compute $a + b$.

Strategy: put $S = x^2 + y^2 + z^2$, bound $|xyz|$ by AM-GM, and read S off a monotone function. [1]

[1] BMT 2018/6

AM-GM: Try it

Let x, y, z be real numbers with $7(x^2 + y^2 + z^2) + 9xyz = 12$. The minimum of $x^2 + y^2 + z^2$ equals a/b in lowest terms. Compute $a + b$.

Hint: Put $S = x^2 + y^2 + z^2$. Bound $|xyz|$ by AM-GM on x^2, y^2, z^2, and check the right side is increasing in S. Where is equality? [2]

[2] BMT 2018/6

AM-GM: Solution — bounding the product

Why? Two unknowns, S and xyz, in one equation. AM-GM on x^2, y^2, z^2 removes xyz:

$$|xyz| = \sqrt{x^2y^2z^2} \leq \left(\frac{x^2+y^2+z^2}{3}\right)^{3/2} = \left(\frac{S}{3}\right)^{3/2}.$$

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Since $xyz \leq |xyz|$, the constraint becomes an inequality in S alone:

$$12 = 7S + 9xyz \leq 7S + 9\left(\frac{S}{3}\right)^{3/2}.$$

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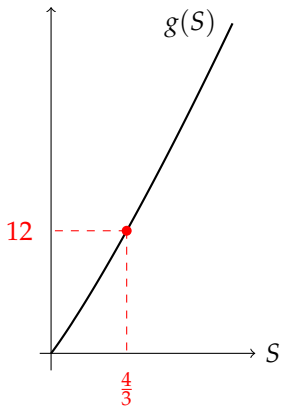
The right side is strictly increasing in $S \ge 0$ (a sum of two increasing terms; its derivative is $7 + \frac{3}{2}\sqrt{3S} > 0$). So it is small exactly when S is small.

AM-GM: Solution — reading off the answer



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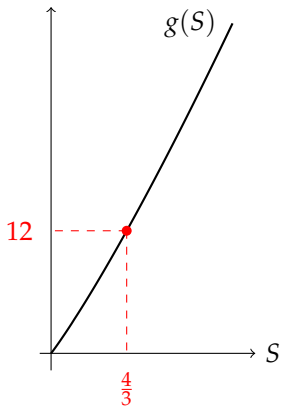

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Where does it hit 12? Guess the symmetric point: $g\left(\frac{4}{3}\right) = \frac{28}{3} + \frac{8}{3} = 12$.

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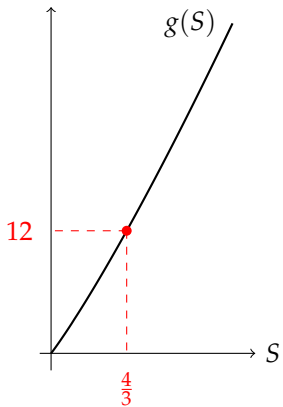

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$12 \le g(S)$ and g increasing force $S \ge \frac{4}{3}$. Equality needs $x^2 = y^2 = z^2$ and $xyz > 0$: $x = y = z = \frac{2}{3}$.

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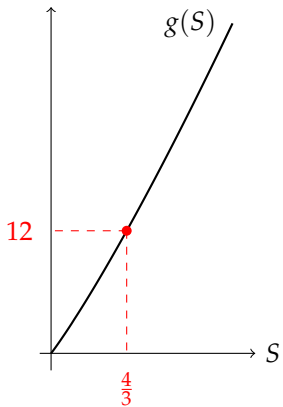

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$\boxed{\min S = \frac{4}{3}}$, so $a + b = 4 + 3 = \boxed{7}$.

Practice Exercises

Two problems to try — the same moves you just saw.

Practice: AM-GM again

Exercise 1. Let x, y, z be real numbers with $x^2 + y^2 + z^2 + xyz = 4$. Find the minimum of $x^2 + y^2 + z^2$.

Hint: Put $S = x^2 + y^2 + z^2$ and bound $|xyz|$ by AM-GM. Where is equality attained?

Practice: split the terms

Exercise 2. Let x, y, and z be real numbers with $xyz = 10$. What is the maximum possible value of $x^3y^3z^3 - 3x^4 - 12y^2 - 12z^4$?

Hint: $x^3y^3z^3$ is fixed, so minimize $3x^4 + 12y^2 + 12z^4$. Split the terms so that each variable's total exponent is the same, then apply AM-GM.

Answers

1. Exercise 1: $\boxed{3}$
2. Exercise 2: $\boxed{760}$