AM-GM Under a Symmetric Constraint

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Bound the product by AM-GM, then read the answer off a monotone function. From $c_1 S + c_2 xyz = C$ with $S = x^2 + y^2 + z^2$: AM-GM gives $|xyz| \le (S/3)^{3/2}$, so the constraint becomes a monotone inequality in S alone — the rigorous version of "by symmetry." Cue: a symmetric constraint plus a symmetric objective, where "by symmetry" feels unearned.

Problem 1 (BMT 2018/6)**.** Let x, y, z be real numbers with $7(x^2 + y^2 + z^2) + 9xyz = 12$. The minimum of $x^2 + y^2 + z^2$ equals a/b in lowest terms. Compute $a + b$.

Solution. Put $S = x^2 + y^2 + z^2$. AM-GM on x^2, y^2, z^2 gives $|xyz| \le (S/3)^{3/2}$, so

$$12 = 7S + 9xyz \le 7S + 9(S/3)^{3/2}.$$

The right side is strictly increasing in $S \ge 0$ (it is a sum of the increasing functions $7S$ and $9(S/3)^{3/2}$; its derivative is $7 + \frac{3}{2}\sqrt{3S} > 0$), and equals 12 at $S = 4/3$:

$$7 \cdot \tfrac{4}{3} + 9\left(\tfrac{4}{9}\right)^{3/2} = \tfrac{28}{3} + \tfrac{8}{3} = 12.$$

If $S < 4/3$, the right side would be less than 12, which is impossible. Hence $S \ge 4/3$, with equality at $x = y = z = 2/3$ (equality in AM-GM needs $x^2 = y^2 = z^2$, and $xyz > 0$; check: $7 \cdot \frac{4}{3} + 9 \cdot \frac{8}{27} = 12$). So $\min S = 4/3$ and $a + b = 4 + 3 = 7$. □

Exercise 2. Let x, y, z be real numbers with $x^2 + y^2 + z^2 + xyz = 4$. Find the minimum of $x^2 + y^2 + z^2$. Hint: 1

Exercise 3. Let x, y, and z be real numbers with $xyz = 10$. What is the maximum possible value of $x^3y^3z^3 - 3x^4 - 12y^2 - 12z^4$? Hint: 2

Answers: 3; 760.

Hints

1. Put $S = x^2 + y^2 + z^2$ and bound $|xyz|$ by AM-GM. Where is equality attained?
2. $x^3y^3z^3$ is fixed, so minimize $3x^4 + 12y^2 + 12z^4$. Split the terms so that each variable's total exponent is the same, then apply AM-GM.