# Minkowski: Make It a Triangle

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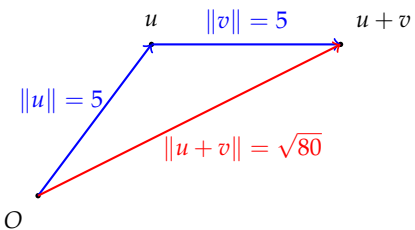


The key idea: turn each root into the length of a vector; the triangle inequality manufactures the cross terms.

$\|u\| + \|v\| \geq \|u + v\|$: with $u = (3, 4), v = (5, 0), 5 + 5 > \sqrt{80}$.

# The Move

*Square roots of quartics begging for cross terms.*

# Minkowski: the move

Write each square root as the length of a vector; the triangle inequality manufactures the cross terms.

**When:** a sum of square roots of sums of squares (often quartics), and the other side has cross terms like $xy^2$.

**The move:** $\sqrt{a^2 + b^2}$ is the length of $(a, b)$, and two sides of a triangle beat the third: $\|u\| + \|v\| \geq \|u + v\|$.

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**The move:** $\sqrt{a^2+b^2}$ is the length of $(a, b)$, and two sides of a triangle beat the third: $\|u\| + \|v\| \geq \|u+v\|$.

- The same root has several vector forms: $\sqrt{x^4+4} = \|(x^2, 2)\| = \|(x^2-2, 2x)\|$ (complete the square).
- Pick the forms so that $\|u+v\|^2$ contains the cross terms you want; then square and compare.
- Equality exactly when $u$ and $v$ point the same way — that finds the extreme case.

# Worked Example

*Choose the components so that $\|u + v\|^2$ produces the cross terms you want.*

# Minkowski: Example

Compute the smallest real number $C$ such that
$x^2(1+y)+y^2(1+x) \leq \sqrt{(x^4+4)(y^4+4)}+C$ for all real $x, y$.

*Strategy:* write each root as a vector norm, apply the triangle inequality, square and compare. [1]

[1] BMT 2020/8

# Minkowski: Try it

Compute the smallest real number $C$ such that
$x^2(1+y)+y^2(1+x) \leq \sqrt{(x^4+4)(y^4+4)} + C$ for all real $x, y$.

*Hint:* Write $\sqrt{x^4+4} = \|u\|$, $\sqrt{y^4+4} = \|v\|$ with $u = (x^2-2, 2x)$, $v = (2, y^2)$.
What is $u+v$? Square the triangle inequality and compare. [2]

[2] BMT 2020/8

# Minkowski: Solution — the vectors

*Why vectors?* Squaring $\|u\| + \|v\|$ gives $2\|u\|\|v\| = 2\sqrt{(x^4+4)(y^4+4)}$ — exactly the root in the problem.

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*Which vectors?* We want the cross term $xy^2$; it appears in $(2x + y^2)^2$, so $2x$ and $y^2$ must share a coordinate:

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Their sum: $u + v = (x^2,\, 2x + y^2)$, so $\|u+v\|^2 = x^4 + (2x+y^2)^2$ — the cross term $4xy^2$ is there.

# Minkowski: Solution — squaring and comparing

Square $\|u\| + \|v\| \geq \|u + v\|$ (both sides are $\geq 0$, so squaring is safe):

$$x^4 + y^4 + 8 + 2\sqrt{(x^4+4)(y^4+4)} \geq x^4 + (2x + y^2)^2.$$

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$$8 + 2\sqrt{(x^4+4)(y^4+4)} \geq 4x^2 + 4xy^2 \iff x^2 + xy^2 \leq 2 + \tfrac{1}{2}\sqrt{(x^4+4)(y^4+4)}.$$

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The target is symmetric, so add the copy with $x$ and $y$ swapped:

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So $C = 4$ works. *Is it sharp?* Equality needs $u \parallel v$: at $x = y = 2$, $u = v = (2,4)$, and the left side is $24 = 20 + 4$. $\boxed{C = 4}$.

# Practice Exercises

*One problem to try — the same move you just saw.*

# Practice: parallel vectors

**Exercise 1.** Find the minimum of $\sqrt{x^4+4} + \sqrt{(2-x)^4+4}$ over real $x$.

*Hint:* Write each square root as the norm of a vector in $\mathbb{R}^2$, e.g. $\sqrt{x^4+4} = \|(x^2, 2)\|$; their sum is at least the norm of the sum. When are the two vectors parallel and pointing the same way?

# Answers

1. Exercise 1: $\boxed{2\sqrt{5}}$