Minkowski: Make It a Triangle

David Liu

Minkowski: write each square root as the length of a vector, then use the triangle inequality. A root of a sum of two squares is a length: $\sqrt{x^4+4} = \|(x^2, 2)\|$, or, completing the square, $\sqrt{x^4+4} = \sqrt{(x^2-2)^2+(2x)^2} = \|(x^2-2, 2x)\|$. The triangle inequality $\|u\| + \|v\| \geq \|u+v\|$ manufactures exactly the cross terms you need; choose the components so that $\|u+v\|^2$ contains them, then square both sides and compare. Equality holds when u and v point the same way. Cue: square roots of quartics begging for cross terms.

Problem 1 (BMT 2020/8)**.** Compute the smallest real number C such that $x^2(1+y)+y^2(1+x) \leq \sqrt{(x^4+4)(y^4+4)} + C$ for all real x, y.

Solution. Write $\sqrt{x^4+4} = \|u\|$ and $\sqrt{y^4+4} = \|v\|$ with $u = (x^2-2, 2x)$ and $v = (2, y^2)$, so $u+v = (x^2, 2x+y^2)$. (We put $2x$ and y^2 in the same coordinate because the cross term xy^2 we want comes from $(2x+y^2)^2$.) Then $\|u\| + \|v\| \geq \|u+v\|$, and squaring,

$$x^4+4+y^4+4+2\sqrt{(x^4+4)(y^4+4)} \geq x^4+(2x+y^2)^2.$$

Expanding $(2x+y^2)^2 = 4x^2+4xy^2+y^4$ and cancelling x^4+y^4 leaves $8+2\sqrt{(x^4+4)(y^4+4)} \geq 4x^2+4xy^2$, that is, $x^2+xy^2 \leq 2+\frac{1}{2}\sqrt{(x^4+4)(y^4+4)}$. Adding the symmetric copy (swap x and y),

$$x^2(1+y)+y^2(1+x) \leq 4+\sqrt{(x^4+4)(y^4+4)},$$

so $C = 4$ works. At $x = y = 2$ the left side is 24 and the square root is 20, so C cannot be smaller. (This is the equality case: at $x = y = 2$, $u = v = (2,4)$ point the same way.) Hence $C = 4$. □

Exercise 2. Find the minimum of $\sqrt{x^4+4} + \sqrt{(2-x)^4+4}$ over real x. Hint: 1

Answers: $2\sqrt{5}$.

Hints

1. Write each square root as the norm of a vector in $\mathbb{R}^2$, e.g. $\sqrt{x^4+4} = \|(x^2, 2)\|$; their sum is at least the norm of the sum. When are the two vectors parallel and pointing the same way?