Interpolation in the Binomial Basis

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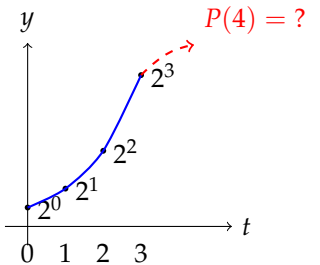
The key idea: match the prescribed values with a low-degree interpolant, then repair the degree with a multiple of $t(t-1)\cdots(t-n)$.



P monic of degree 4, $P(k) = 2^k$ at $k = 0, 1, 2, 3$ — what is $P(4)$?

Binomial-Basis Interpolation

Match the prescribed values with a low-degree interpolant, then repair the degree with a multiple of $t(t-1)\cdots(t-n)$.

The technique in two sentences

When to reach for it: a polynomial's values are prescribed at consecutive integers $0, 1, \ldots, n$, and you are asked for its value somewhere else.

The move:

1. **Match.** Write down a polynomial S of degree $\leq n$ that takes the prescribed values — in the basis $\binom{t}{i}$ this is easy.
2. **Repair.** $P - S$ vanishes at $0, 1, \ldots, n$, so $P - S = c \cdot t(t-1)\cdots(t-n)$ (times a polynomial, if $\deg P$ is larger); the leading coefficient of P fixes c.

Then evaluate wherever you like.

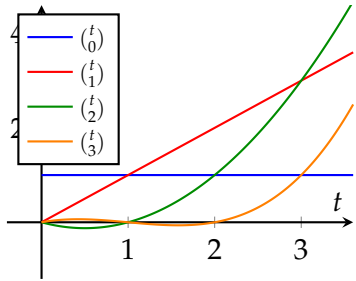
Binomial-basis interpolation: the setup

Every polynomial of degree at most n has a unique expansion

$$P(t) = a_0\binom{t}{0} + a_1\binom{t}{1} + \cdots + a_n\binom{t}{n}.$$

Why this basis: at an integer k, $\binom{k}{i} = 0$ for $i > k$ — each value $P(k)$ pins down the coefficients one at a time.
And the binomial theorem hands us the interpolant for 2^k:

$$\sum_{i=0}^{n}\binom{k}{i} = 2^k \qquad (\text{integers } 0 \le k \le n).$$



$\binom{t}{i}$ vanishes at $t = 0, 1, \ldots, i-1$.

Warm-up: the title-slide puzzle

P is monic of degree 4 and $P(k) = 2^k$ for $k = 0, 1, 2, 3$. Find $P(4)$.

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Match. $S(t) = \binom{t}{0} + \binom{t}{1} + \binom{t}{2} + \binom{t}{3}$ has $S(k) = \sum_{i \le k} \binom{k}{i} = 2^k$ for $k = 0, \ldots, 3$.
Why: the binomial theorem is exactly a formula for 2^k as a sum of $\binom{k}{i}$.

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Repair. $P - S$ is monic of degree 4 (since $\deg S = 3$) with roots $0, 1, 2, 3$, so

$$P(t) = S(t) + t(t-1)(t-2)(t-3).$$

Why: a monic quartic with four known roots is completely determined.

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Evaluate. $P(4) = (1 + 4 + 6 + 4) + 4! = 15 + 24 = \boxed{39}$.

Binomial-basis interpolation: Example

Given $x_1 \cdots x_{2022} = 1$, $(x_1 + 1) \cdots (x_{2022} + 1) = 2, \ldots,$
$(x_1 + 2021) \cdots (x_{2022} + 2021) = 2^{2021}$, compute $(x_1 + 2022) \cdots (x_{2022} + 2022)$.

Strategy: package the products as one polynomial P, match its values with a binomial-basis sum, then repair the degree. [1]

[1] BMT 2022/8

Binomial-basis interpolation: Try it

Given $x_1 \cdots x_{2022} = 1$, $(x_1 + 1) \cdots (x_{2022} + 1) = 2, \ldots,$
$(x_1 + 2021) \cdots (x_{2022} + 2021) = 2^{2021}$, compute $(x_1 + 2022) \cdots (x_{2022} + 2022)$.

Hint: Set $P(t) = \prod_{i=1}^{2022}(x_i + t)$. What are $P(0), P(1), \ldots, P(2021)$? Which low-degree sum takes exactly those values — and what multiple of $t(t-1)\cdots(t-2021)$ repairs the degree? [2]

[2] BMT 2022/8

Binomial-basis interpolation: Solution (match)

Package. $P(t) = \prod_{i=1}^{2022}(x_i + t)$ is monic of degree 2022, with $P(k) = 2^k$ for $k = 0, 1, \ldots, 2021$.
Why: the 2022 given products are the values of one polynomial at consecutive integers — exactly the cue.

Binomial-basis interpolation: Solution (match)

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Match. The sum

$$S(t) = \sum_{i=0}^{2021} \binom{t}{i}$$

interpolates the same values: at an integer $k \leq 2021$ the terms with $i > k$ vanish, so $S(k) = \sum_{i=0}^{k} \binom{k}{i} = 2^k$.
Why: the values are powers of 2, and the binomial theorem builds 2^k from $\binom{k}{i}$.

Binomial-basis interpolation: Solution (repair and evaluate)

Repair. $P - S$ vanishes at the 2022 points $0, 1, \ldots, 2021$, and $\deg(P - S) \leq 2022$, so

$$P(t) - S(t) = c \cdot t(t-1) \cdots (t-2021).$$

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P is monic and $\deg S = 2021$, so comparing t^{2022} coefficients gives $c = 1$.
Why: S only fixes the values; the leading coefficient is what pins down P.

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Evaluate at $t = 2022$, where the product is 2022! and the sum misses only $\binom{2022}{2022}$:

$$P(2022) = 2022! + \sum_{i=0}^{2021} \binom{2022}{i} = 2022! + \left(2^{2022} - 1\right).$$

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$$\boxed{P(2022) = 2022! + 2^{2022} - 1}.$$

Practice Exercises

Two problems to try — the same move you just saw.

Practice: binomial-basis repair

Exercise 1. Let P be monic of degree 3 with $P(0) = 1$, $P(1) = 2$, $P(2) = 4$. Compute $P(3)$.

Hint: $S(t) = \binom{t}{0} + \binom{t}{1} + \binom{t}{2}$ matches the three values; $P - S$ is monic of degree 3 with roots $0, 1, 2$.

Practice: line repair

Exercise 2. Find the sum of the coefficients of $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$.

Hint: The values $11, 17, 23$ lie on the line $y = x + 6$; study $P(x) - (x + 6)$ and repair the degree with a multiple of $(x - 5)(x - 11)(x - 17)$. The x^3 coefficient pins down the missing factor, and the sum of the coefficients is $P(1)$. [3]

[3] HMMT 2011 Team #3

Answers

1. Exercise 1: $\boxed{13}$
2. Exercise 2: $\boxed{-3193}$