Interpolation in the Binomial Basis

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Values of a polynomial at $0, 1, \ldots, n$ prescribed? Match them with a sum of binomial polynomials $\binom{t}{i}$, then repair the degree with a multiple of $t(t-1)\cdots(t-n)$.

Suppose a polynomial takes prescribed values at several integer points — say $P(k) = 2^k$ for $k = 0, \ldots, 2021$ — and you need its value somewhere else, like $P(2022)$. Rather than solving for the coefficients of P directly, we *interpolate*: we rebuild the polynomial from its values using a *basis*, a set of building blocks with which every polynomial of bounded degree can be written in exactly one way. Today's building blocks are the binomial polynomials $\binom{t}{i} = \frac{t(t-1)\cdots(t-i+1)}{i!}$: at an integer k, all but the first $k+1$ of them vanish, so the values pin down the coefficients one at a time, and we finish by repairing the degree with a multiple of $t(t-1)\cdots(t-n)$. We work one example fully, then you try the exercises.

Technique: Interpolate in the binomial basis

Every polynomial P of degree at most n has a unique expansion

$$P(t) = a_0\binom{t}{0} + a_1\binom{t}{1} + \cdots + a_n\binom{t}{n}, \qquad \binom{t}{i} = \frac{t(t-1)\cdots(t-i+1)}{i!}.$$

Why this basis: at an integer k, $\binom{k}{i} = 0$ for $i > k$, so $P(k)$ sees only $a_0, \ldots, a_k$ — the values pin down the coefficients one at a time. And the binomial theorem says $\sum_{i=0}^{n}\binom{k}{i} = 2^k$ for integers $0 \le k \le n$. The master move: match the prescribed values with a low-degree interpolant, then repair the degree with a multiple of $t(t-1)\cdots(t-n)$. Recognition cue: when the prescribed values are 2^k at $0, \ldots, n$, the binomial theorem hands you the interpolant.

Problem 1 (BMT 2022/8)**.** Given

$$x_1 \cdots x_{2022} = 1, \quad (x_1+1)\cdots(x_{2022}+1) = 2, \quad \ldots, \quad (x_1+2021)\cdots(x_{2022}+2021) = 2^{2021},$$

compute $(x_1+2022)\cdots(x_{2022}+2022)$.

Solution. Set $P(t) = \prod_{i=1}^{2022}(x_i + t)$: monic of degree 2022, with $P(k) = 2^k$ for $k = 0, 1, \ldots, 2021$. The sum

$$S(t) = \sum_{i=0}^{2021}\binom{t}{i}$$

interpolates the same values: at integer $k \le 2021$, the terms with $i > k$ vanish, so $S(k) = \sum_{i=0}^{k}\binom{k}{i} = 2^k$.

Thus $P - S$ vanishes at the 2022 points $0, 1, \ldots, 2021$. Since $\deg(P - S) \leq 2022$,

$$P(t) - S(t) = c \cdot t(t-1) \cdots (t-2021)$$

for some constant c. Both P and the product are monic while $\deg S = 2021$, so comparing leading coefficients gives $c = 1$:

$$P(t) = t(t-1) \cdots (t-2021) + \sum_{i=0}^{2021} \binom{t}{i}.$$

Evaluate at $t = 2022$:

$$P(2022) = 2022! + \sum_{i=0}^{2021} \binom{2022}{i} = 2022! + \left(2^{2022} - \binom{2022}{2022}\right) = 2022! + 2^{2022} - 1.$$

□

Exercise 2. Let P be monic of degree 3 with $P(0) = 1$, $P(1) = 2$, $P(2) = 4$. Compute $P(3)$. Hint: 1

Exercise 3 (HMMT 2011 Team #3)**.** Find the sum of the coefficients of $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$. Hint: 2

Answers: 13; -3193.

Hints

1. $S(t) = \binom{t}{0} + \binom{t}{1} + \binom{t}{2}$ matches the three values; $P - S$ is monic of degree 3 with roots $0, 1, 2$.
2. The values $11, 17, 23$ lie on the line $y = x + 6$; study $P(x) - (x + 6)$ and repair the degree with a multiple of $(x - 5)(x - 11)(x - 17)$. The x^3 coefficient pins down the missing factor, and the sum of the coefficients is $P(1)$.