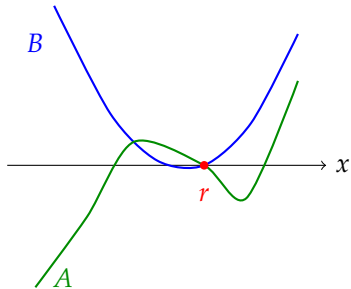
A Common Root Kills Every Remainder

David Liu

The key idea: if r kills both A and B, it kills every remainder of A divided by B — divide, and read the equations off the remainder.



$$A(r) = B(r) = 0 \implies R(r) = 0$$

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The technique in two sentences

When to reach for it: two polynomial equations are said to share a root, the root itself is ugly or unknown, and the question is about a *parameter* in the coefficients.

The move: divide one polynomial by the other (or just subtract them). Anything that kills both also kills the remainder, which has lower degree — so the common root is easy to read off it, and plugging it back gives an equation for the parameter.

Related move: if $f(a) = f(b)$, then $f(x) - f(a)$ has roots a and b, so $(x-a)(x-b)$ factors out — and the quotient is a smaller polynomial to study.

Move 1: A Common Root Kills Every Remainder

Two polynomials share an unseen root — divide one by the other and force the remainder to admit it.

Move 1: the remainder trick

If r is a root of both A and B, dividing A by B leaves a remainder R that also vanishes at r. (Why: $A = QB + R$, so $R(r) = A(r) - Q(r)B(r) = 0$.)

So instead of finding r, force the remainder's coefficients to admit r — the coefficients give equations directly.

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$$(x^2 + kx + 1) - (x^2 + x + k) = (k-1)x + (1-k) = (k-1)(x-1).$$

Why subtract: both are monic quadratics, so subtracting is one step of division and kills x^2. Then plug the forced root $x = 1$ back in.

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The common root must kill the remainder:

$k = 1$ (the polynomials coincide), or the root is 1,

$$1 + 1 + k = 0 \implies \boxed{k = -2}.$$

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Move 2: Equal Values Factor Out

f takes the same value twice — factor $(x-a)(x-b)$ out of $f(x)-f(a)$ and study the smaller quotient.

Move 2: Example

A polynomial f satisfies $f(-2) = 3, f(-1) = -3 = f(1), f(2) = 6, f(3) = 5$. What is the minimum possible degree of f?

Strategy: $f(-1) = f(1)$ — factor $(x+1)(x-1)$ out of $f(x) + 3$, then count what the quotient must do. [1]

[1] BMT 2018/3

Move 2: Try it

A polynomial f satisfies $f(-2) = 3, f(-1) = -3 = f(1), f(2) = 6, f(3) = 5$. What is the minimum possible degree of f?

Hint: Write $f(x) = (x^2 - 1)g(x) - 3$. What values must g take at $-2, 2, 3$ — and can three non-collinear points lie on a line? [2]

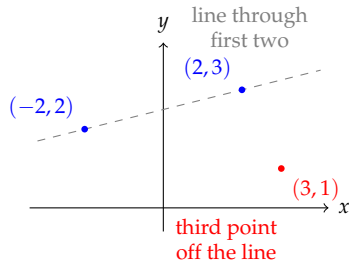
[2] BMT 2018/3

Move 2: Solution

From $f(-1) = f(1) = -3$: $f(x) + 3$ vanishes at $x = \pm 1$, so

$$f(x) = (x^2 - 1)g(x) - 3, \qquad \deg g = \deg f - 2.$$

Why: equal values become common roots, and we can factor them out.



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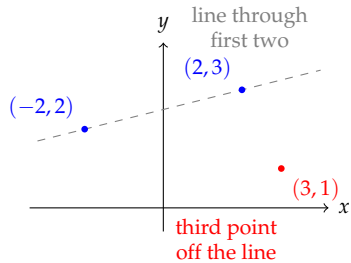

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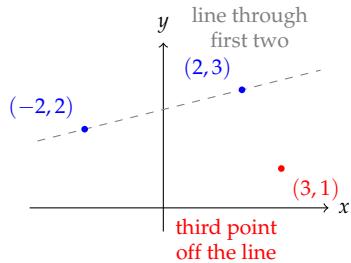

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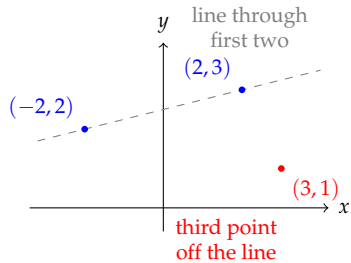
Hence $\deg f \geq 4$, achieved by the quadratic interpolant:

$$\boxed{\text{minimum degree } = 4}.$$



Practice Exercises

Two problems to try — one by division, one by counting roots.

Practice: Move 1 (division)

Exercise 1. Find the product of all complex numbers d such that $x^3 + 2x^2 + 3x + 4 = 0$ and $x^2 + dx + 3 = 0$ have a common root.

Hint: Divide the cubic by the quadratic: $x^3 + 2x^2 + 3x + 4 = (x + 2 - d)(x^2 + dx + 3) + d(d-2)x + (3d - 2)$. The common root x annihilates the remainder ($d = 0, 2$ fail), so $x = \frac{2-3d}{d^2-2d}$; substitute into $x^2 + dx + 3 = 0$ to get a cubic in d. Every complex root counts — use Vieta for the product. [3]

[3] BMT 2018/5

Practice: counting roots

Exercise 2. P is monic of degree 2021 with real coefficients, meeting the x-axis exactly at $(1,0),\ldots,(2020,0)$ and nowhere else. The mean of all possible values of $P(2021)$ equals $a!/b$ for positive integers a,b with a minimal. Compute $a+b$.

Hint: $1,\ldots,2020$ use up 2020 of the 2021 roots, leaving one. Non-real roots come in conjugate pairs, so the leftover root is real, and "nowhere else" forces it to repeat one of $1,\ldots,2020$. If the repeated root is r, then $P(2021)=(2021-r)\cdot 2020!$; average over the choices of r. [4]

[4] BMT 2020/3

Answers

1. Exercise 1: $\boxed{1}$
2. Exercise 2: $\boxed{2023}$