A Common Root Kills Every Remainder

David Liu

Two polynomials share a root you cannot find? Divide one by the other: the common root must also kill the remainder, and the remainder's coefficients hand you equations.

Two polynomials can share a root you never see: in the first exercise below, $x^3+2x^2+3x+4=0$ and $x^2+dx+3=0$ have a common root r, and the question asks about the unknown parameter d, not about r.

If r is a root of both A and B, dividing A by B leaves a remainder R that also vanishes at r. So instead of finding r, force the remainder's coefficients to admit r — the coefficients give equations directly. This is the heart of the Euclidean algorithm for polynomials.

Mini-example: for which k do x^2+kx+1 and x^2+x+k share a root? Subtracting gives the remainder $(k-1)x+(1-k)=(k-1)(x-1)$; the common root must kill it, so either $k=1$ (the polynomials coincide) or the common root is 1, forcing $k=-2$.

A related move: if $P(a)=P(b)$ for $a\neq b$, then $P(x)-P(a)$ vanishes at both a and b — factor out $(x-a)(x-b)$ and study the smaller quotient.

Problem 1 (BMT 2018/3)**.** A polynomial f satisfies $f(-2)=3$, $f(-1)=-3=f(1)$, $f(2)=6$, $f(3)=5$. What is the minimum possible degree of f?

Solution. From $f(-1)=f(1)=-3$, $f(x)+3$ vanishes at $x=\pm1$, so $f(x)=(x^2-1)g(x)-3$ with $\deg g=\deg f-2$. Then $g(-2)=2$, $g(2)=3$, $g(3)=1$; the three points are not collinear, so $\deg g\geq 2$ and $\deg f\geq 4$, achieved by the quadratic interpolant. □

Exercise 2 (BMT 2018/5)**.** Find the product of all complex numbers d such that $x^3+2x^2+3x+4=0$ and $x^2+dx+3=0$ have a common root. Hint: 1

Exercise 3 (BMT 2020/3)**.** P is monic of degree 2021 with real coefficients, meeting the x-axis exactly at $(1,0),\ldots,(2020,0)$ and nowhere else. The mean of all possible values of $P(2021)$ equals $a!/b$ for positive integers a,b with a minimal. Compute $a+b$. Hint: 2

Answers: 1; 2023.

Hints

1. Divide the cubic by the quadratic: $x^3+2x^2+3x+4=(x+2-d)(x^2+dx+3)+d(d-2)x+(3d-2)$. The common root x annihilates the remainder. ($d=0$ and $d=2$ leave a nonzero constant remainder, so they fail.) Hence $x=\frac{2-3d}{d^2-2d}$; substitute into $x^2+dx+3=0$ and clear

denominators to get a cubic in d. Every complex root of that cubic counts, so Vieta gives the product: for $ad^3 + bd^2 + cd + e$, the three roots multiply to $-e/a$.

2. $1, \ldots, 2020$ use up 2020 of the 2021 roots (with multiplicity), leaving one more. Non-real roots of a real polynomial come in conjugate pairs, so a single leftover root is real; "nowhere else" then forces it to repeat one of $1, \ldots, 2020$. If the repeated root is r, then $P(2021) = (2021 - r) \cdot 2020!$; average over the choices of r.