Shift-and-Multiply for Recurrences

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The key idea: recurrence $\times$ geometric weight, summed — shifting peels off finitely many terms, and the tail is still S.

$$\begin{aligned} & \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots = S - a_0 - \frac{a_1}{10} \\ + \quad & \frac{a_1}{10^2} + \frac{a_2}{10^3} + \cdots = \frac{1}{10}(S - a_0) \\ + \quad & \frac{xa_0}{10^2} + \frac{xa_1}{10^3} + \cdots = \frac{x}{100}S \end{aligned}$$

Adding the rows: the recurrence becomes one equation for S itself.

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The Shift-and-Multiply Move

Recurrence $\times$ geometric weight, summed: index shifts become multiplications of S.

Shift-and-multiply: the setup

Recurrence $\times$ geometric weight, summed: index shifts become multiplications of S.

When to reach for it: a sequence known only through a recurrence, and a weighted infinite sum $S = \sum a_n r^n$ to evaluate. **The move:** multiply the recurrence by r^n (shifted to match), add all the equations, and solve for S.

Computing terms one by one is hopeless — work with the whole series at once. Three mechanics:

- Choose the weight's shift to match the highest index: if the recurrence reaches a_{n+2}, multiply the n-th equation by $10^{-(n+2)}$.
- Shifting peels off finitely many leading terms: $\sum_{n\geq 0} a_{n+2}/10^{n+2}$ is S minus $a_0 + a_1/10$.
- The tail is still S: every sum rewrites in terms of S, so S satisfies a plain algebraic equation — solve it.

Shift-and-multiply: Example

Let $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + xa_n$ for $n \geq 0$, where x is real. Given $\sum_{n=0}^{\infty} \frac{a_n}{10^n} = 1$, compute x.

Strategy: multiply the recurrence by the geometric weight, sum, and rewrite every sum in terms of S. [1]

[1] BMT 2023/6

Shift-and-multiply: try it

Let $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + xa_n$ for $n \geq 0$, where x is real. Given $\sum_{n=0}^{\infty} \frac{a_n}{10^n} = 1$, compute x.

Hint: The recurrence reaches a_{n+2} — what should multiply the n-th equation? After summing, what is $\sum_{n \geq 0} a_{n+2}/10^{n+2}$ in terms of S? [2]

[2] BMT 2023/6

Shift-and-multiply: Solution

Step 1. Multiply the n-th equation by $10^{-(n+2)}$ and sum over $n \geq 0$. *Why* $10^{-(n+2)}$*: it matches the top index* a_{n+2}*, so the left side is a tail of* S*.*

$$\sum_{n\geq 0} \frac{a_{n+2}}{10^{n+2}} = \sum_{n\geq 0} \frac{a_{n+1}}{10^{n+2}} + x \sum_{n\geq 0} \frac{a_n}{10^{n+2}}.$$

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Step 2. Rewrite every sum through S — each is S with a few leading terms peeled off, times a constant:

$$\text{LHS} = S - a_0 - \tfrac{a_1}{10}, \qquad \tfrac{1}{10} \sum_{n\geq 0} \frac{a_{n+1}}{10^{n+1}} = \tfrac{1}{10}(S - a_0), \qquad \tfrac{x}{100} \sum_{n\geq 0} \frac{a_n}{10^n} = \tfrac{x}{100} S.$$

Shift-and-multiply: Solution

Step 1. Multiply the n-th equation by $10^{-(n+2)}$ and sum over $n \geq 0$. *Why* $10^{-(n+2)}$*: it matches the top index* a_{n+2}*, so the left side is a tail of* S*.*

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Step 2. Rewrite every sum through S — each is S with a few leading terms peeled off, times a constant:

$$\text{LHS} = S - a_0 - \tfrac{a_1}{10}, \qquad \tfrac{1}{10} \sum_{n\geq 0} \frac{a_{n+1}}{10^{n+1}} = \tfrac{1}{10}(S - a_0), \qquad \tfrac{x}{100} \sum_{n\geq 0} \frac{a_n}{10^n} = \tfrac{x}{100} S.$$

Step 3. Now the only unknown is x. With $a_0 = 0$, $a_1 = 1$, $S = 1$:

$$1 - \tfrac{1}{10} = \tfrac{1}{10} + \tfrac{x}{100} \quad \Longrightarrow \quad \tfrac{8}{10} = \tfrac{x}{100}, \qquad \boxed{x = 80}.$$

Shift-and-multiply: Solution

Step 1. Multiply the n-th equation by $10^{-(n+2)}$ and sum over $n \geq 0$. *Why* $10^{-(n+2)}$*: it matches the top index* a_{n+2}*, so the left side is a tail of* S*.*

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Step 2. Rewrite every sum through S — each is S with a few leading terms peeled off, times a constant:

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Sanity check: for $x = 80$, $t^2 = t + 80$ has roots $\frac{1 \pm \sqrt{321}}{2} \approx 9.46$, -8.46, smaller than 10 in size, so $a_n/10^n \to 0$ geometrically and the series converges.

Practice Exercises

Two problems to try — the same shift-and-multiply move.

Practice: shift-and-multiply

Exercise 1. Let $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$. Compute $\sum_{n=0}^{\infty} \frac{a_n}{3^n}$.

Hint: Multiply the recurrence by $3^{-(n+2)}$ and sum over $n \geq 0$. The left side is $S - a_0 - a_1/3$; solve the resulting linear equation for S.

Practice: foury vs. fourier

Exercise 2. Let $0 < x < 0.1$. The "foury" series is $4 + 4x + 4x^2 + \cdots$ and the "fourier" series is $4 + 44x + 444x^2 + \cdots$. If the fourier sum is four times the foury sum, compute x.

Hint: The fourier coefficients $t_0, t_1, t_2, \cdots = 4, 44, 444, \ldots$ satisfy $t_{n+1} = 10t_n + 4$. Multiply this recurrence by x^n, sum over $n \geq 0$, and solve the resulting equation for $S = \sum t_n x^n$. [3]

[3] HMMT 2022 General #4

Answers

1. Exercise 1: $\boxed{\frac{3}{4}}$

2. Exercise 2: $\boxed{\frac{3}{40}}$