Shift-and-Multiply for Recurrences

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Recurrence × geometric weight, summed: index shifts become multiplications of S.

Suppose a sequence is given only by a recurrence, for example $a_{n+2} = a_{n+1} + xa_n$, and you need the value of a weighted infinite sum like $\sum a_n/10^n$. Computing the terms one by one is hopeless: the trick is to work with the whole series $S = \sum a_n r^n$ at once. Multiply the recurrence by the geometric weight r^n and add up all the equations; each shifted copy of the recurrence recreates the same series S, so S ends up satisfying a plain algebraic equation you can solve. The pattern: recurrence × geometric weight, summed, turns index shifts into multiplications of S.

Problem 1 (BMT 2023/6)**.** Let $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + xa_n$ for $n \geq 0$, where x is real. Given $\sum_{n=0}^{\infty} \frac{a_n}{10^n} = 1$, compute x.

Solution. Let $S = \sum_{n\geq 0} a_n/10^n = 1$. Multiply the recurrence by $10^{-(n+2)}$ and sum over $n \geq 0$:

$$\sum_{n\geq 0} \frac{a_{n+2}}{10^{n+2}} = \sum_{n\geq 0} \frac{a_{n+1}}{10^{n+2}} + x \sum_{n\geq 0} \frac{a_n}{10^{n+2}}.$$

The left side is $S - a_0 - a_1/10$. The first sum on the right is $\frac{1}{10}(S - a_0)$, and the second is $\frac{x}{100}S$. With $a_0 = 0$, $a_1 = 1$, $S = 1$:

$$1 - \tfrac{1}{10} = \tfrac{1}{10} + \tfrac{x}{100}, \qquad \text{i.e.} \qquad \tfrac{8}{10} = \tfrac{x}{100}.$$

Hence $x = 80$.

Check that the series converges. With $x = 80$ the recurrence is $a_{n+2} = a_{n+1} + 80a_n$, whose characteristic equation $t^2 = t + 80$ has roots $\frac{1\pm\sqrt{321}}{2} \approx 9.46$ and -8.46. So a_n grows roughly like 9.46^n, slower than 10^n, and $\sum a_n/10^n$ really converges; the algebra above was legitimate. □

Exercise 2. Let $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$. Compute $\sum_{n=0}^{\infty} \frac{a_n}{3^n}$. Hint: 1

Exercise 3 (HMMT 2022 General #4)**.** Let $0 < x < 0.1$. The "foury" series is $4 + 4x + 4x^2 + \cdots$ and the "fourier" series is $4 + 44x + 444x^2 + \cdots$. If the fourier sum is four times the foury sum, compute x. Hint: 2

Answers: $\frac{3}{4}$; $\frac{3}{40}$.

Hints

1. Multiply the recurrence by $3^{-(n+2)}$ and sum over $n \geq 0$. The left side is $S - a_0 - a_1/3$; solve the resulting linear equation for S.

2. The fourier coefficients $t_0, t_1, t_2, \cdots = 4, 44, 444, \ldots$ satisfy $t_{n+1} = 10t_n + 4$. Multiply this recurrence by x^n, sum over $n \geq 0$, and solve the resulting equation for $S = \sum t_n x^n$.