Telescoping by Manufactured Differences

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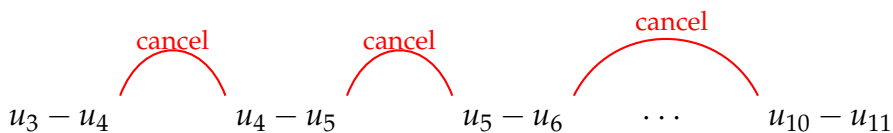
The key idea: write each term as a difference, $a_n = u_n - u_{n+1}$ — then everything in the middle cancels.



Only the first and last terms survive: $\sum(u_n - u_{n+1}) = u_{\text{first}} - u_{\text{last}}$.

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Manufactured Differences

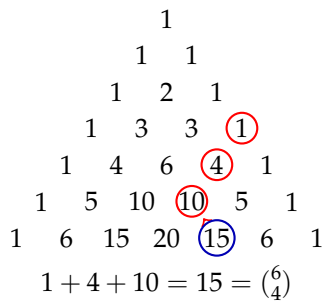
Sums that don't obviously collapse: build the difference yourself.

Telescoping: the setup

Make the difference yourself. **When:** a sum that does not obviously collapse. **The move:** rewrite each term as $u_n - u_{n+1}$, then only the ends survive. Two ways to build u_n:

- Numerator = difference of the denominator's factors: $\dfrac{B-A}{AB} = \dfrac{1}{A} - \dfrac{1}{B}$.
- Products of consecutive integers $\to$ binomial coefficients, summed by hockey-stick:

$$\sum_{m=r}^{N} \binom{m}{r} = \binom{N+1}{r+1}.$$



Telescoping: Example

With $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the value of $\sum_{n=3}^{10} \frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}}$ equals p/q in lowest terms. Compute $p+q$.

Strategy: the numerator should remind you of Pascal's rule — manufacture the difference, then telescope. [1]

[1] BMT 2020/6

Telescoping: try it

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Hint: Pascal: $\binom{n+1}{3} - \binom{n}{3} =$? Then write the summand as $\frac{B-A}{AB}$. [2]

[2] BMT 2020/6

Telescoping: Solution

Why Pascal? The denominator is a product AB of two neighbouring binomials, so we want the numerator to be $B - A$. By Pascal, $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$ — that's the numerator!

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Split $\frac{B-A}{AB} = \frac{1}{A} - \frac{1}{B}$:

$$\frac{\binom{n+1}{3} - \binom{n}{3}}{\binom{n}{3}\binom{n+1}{3}} = \frac{1}{\binom{n}{3}} - \frac{1}{\binom{n+1}{3}}.$$

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Each term is now $u_n - u_{n+1}$, so it telescopes:

$$\sum_{n=3}^{10} = \frac{1}{\binom{3}{3}} - \frac{1}{\binom{11}{3}} = 1 - \frac{1}{165} = \frac{164}{165}.$$

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$$\boxed{p + q = 164 + 165 = 329}.$$

Shift Differences

The same telescoping works for polynomials at arithmetic progressions.

Shift differences: the setup

For polynomials, the shift-difference drops the degree.

When: a polynomial f of known degree, and you are told several differences $f(x+h) - f(x)$. **The move:** name the shift-difference, kill the known values, telescope.

- For fixed h, $f(x+h) - f(x)$ has degree one lower: the leading terms cancel.
- Subtract a simple polynomial matching the known differences; call the result g. The given points are now *roots* of g, so g is known up to a constant.
- Telescope along steps of size h: $f(x+3h) - f(x)$ is a sum of three differences.

Shift differences: Example

Let f be a polynomial of degree 6 with

$$f(7)-f(1)=1, \quad f(8)-f(2)=16, \quad f(9)-f(3)=81,$$
$$f(10)-f(4)=256, \quad f(11)-f(5)=625.$$

Compute $f(15)-f(-3)$.

Strategy: the differences look like fourth powers — subtract x^4 to get a polynomial g with known roots, then telescope in steps of 6. [3]

[3] BMT 2022/6

Shift differences: try it

Let f be a polynomial of degree 6 with

$$f(7)-f(1)=1, \quad f(8)-f(2)=16, \quad f(9)-f(3)=81,$$
$$f(10)-f(4)=256, \quad f(11)-f(5)=625.$$

Compute $f(15)-f(-3)$.

Hint: $1, 16, 81, 256, 625$ are $1^4, 2^4, 3^4, 4^4, 5^4$. Set $g(x)=f(x+6)-f(x)-x^4$ — what are $\deg g$ and its roots? [4]

[4] BMT 2022/6

Shift differences: Solution

Set $g(x) = f(x+6) - f(x) - x^4$ — *why: subtracting x^4 turns the five given values into zeros.* Since $\deg f = 6$, $\deg\big(f(x+6) - f(x)\big) \leq 5$, so $\deg g \leq 5$.

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The five conditions give $g(1) = \cdots = g(5) = 0$; five roots and degree ≤ 5 leave one unknown constant:

$$g(x) = c(x-1)(x-2)(x-3)(x-4)(x-5).$$

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Telescope, using $f(x+6) - f(x) = g(x) + x^4$ along $-3 \to 3 \to 9 \to 15$:

$$f(15) - f(-3) = \big[g(9) + 9^4\big] + \big[g(3) + 3^4\big] + \big[g(-3) + (-3)^4\big].$$

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$$\boxed{f(15) - f(-3) = 9^4 + 3^4 + 3^4 = 6723}.$$

Practice Exercises

Three problems to try — hockey-stick, manufactured differences, and pairing.

Practice: hockey-stick

Exercise 1. Find $\sum_{i=1}^{2016} i(i+1)(i+2) \pmod{2018}$.

Hint: Write $i(i+1)(i+2) = 6\binom{i+2}{3}$; hockey-stick gives $n(n+1)(n+2)(n+3)/4$. At $n = 2016$ a factor of 2018 appears. [5]

[5] BMT 2018/4

Practice: manufactured differences

Exercise 2. The sum $\sum_{m=1}^{2023} \frac{2m}{m^4+m^2+1}$ equals $\frac{a}{b}$ in lowest terms. Find the remainder when $a+b$ is divided by 1000.

Hint: Factor $m^4+m^2+1 = (m^2-m+1)(m^2+m+1)$ and check that $\frac{2m}{m^4+m^2+1} = \frac{1}{m^2-m+1} - \frac{1}{m^2+m+1}$; the sum telescopes. [6]

[6] PUMaC 2023 Algebra #B2

Practice: pairing

Exercise 3. Define $a_0 = 0$ and $a_n = a_{n-1} + (-1)^n n^2$ for $n \geq 1$. Find a_{100}.

Hint: Unroll the recurrence into a sum, then pair consecutive terms; each pair is a difference of two squares. [7]

[7] BMT 2020 Team #9

Answers

1. Exercise 1: $\boxed{0}$
2. Exercise 2: $\boxed{105}$
3. Exercise 3: $\boxed{5050}$