Telescoping by Manufactured Differences

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No obvious collapse? Manufacture the difference $u_n - u_{n+1}$ yourself, then telescope.

Many contest sums look messy but collapse if you can write each term as a difference, $a_n = u_n - u_{n+1}$, so that consecutive terms cancel: this is called telescoping. The technique here is for sums that do not obviously collapse: you build the difference yourself. One way is to split a fraction whose denominator is a product into a difference of its factors, as in $\frac{\binom{n+1}{3}-\binom{n}{3}}{\binom{n}{3}\binom{n+1}{3}} = \frac{1}{\binom{n}{3}} - \frac{1}{\binom{n+1}{3}}$. The other way handles polynomial summands: write them using binomial coefficients — for instance $i(i+1)(i+2) = 6\binom{i+2}{3}$ — and apply the hockey-stick identity $\sum_{m=r}^{N}\binom{m}{r} = \binom{N+1}{r+1}$ to each piece (with $m = i+2$, $\sum_{i=1}^{n}\binom{i+2}{3} = \binom{n+3}{4}$).

Problem 1 (BMT 2020/6)**.** With $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the value of $\sum_{n=3}^{10}\frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}}$ equals p/q in lowest terms. Compute $p+q$.

Solution. By Pascal, $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$. Hence the summand is

$$\frac{\binom{n+1}{3}-\binom{n}{3}}{\binom{n}{3}\binom{n+1}{3}} = \frac{1}{\binom{n}{3}} - \frac{1}{\binom{n+1}{3}},$$

which telescopes:

$$\sum_{n=3}^{10} = \frac{1}{\binom{3}{3}} - \frac{1}{\binom{11}{3}} = 1 - \frac{1}{165} = \frac{164}{165}.$$

So $p+q = 164+165 = 329$. □

The same telescoping works for *polynomials* at arithmetic progressions: for fixed h, the shift-difference $f(x+h) - f(x)$ has degree one lower (leading terms cancel). If you know its values at several points, subtract a polynomial matching them; those points become roots of the difference, which pins it down up to a constant. Then telescope along steps of size h.

Problem 2 (BMT 2022/6)**.** Let f be a polynomial of degree 6 with

$$f(7) - f(1) = 1, \quad f(8) - f(2) = 16, \quad f(9) - f(3) = 81,$$
$$f(10) - f(4) = 256, \quad f(11) - f(5) = 625.$$

Compute $f(15) - f(-3)$.

Solution. Set $g(x) = f(x+6) - f(x) - x^4$. Since $\deg f = 6$, $\deg\big(f(x+6) - f(x)\big) \le 5$ and $\deg g \le 5$. The five conditions give $g(1) = \cdots = g(5) = 0$, so $g(x) = c(x-1)(x-2)(x-3)(x-4)(x-5)$ for some constant c. Since $f(x+6) - f(x) = g(x) + x^4$, step $-3 \to 3 \to 9 \to 15$ in jumps of 6:

$$f(15) - f(-3) = \big[g(9) + 9^4\big] + \big[g(3) + 3^4\big] + \big[g(-3) + (-3)^4\big].$$

Now $g(9) = c \cdot 6720$, $g(-3) = c \cdot (-6720)$, $g(3) = 0$; the c terms cancel. Hence $f(15) - f(-3) = 9^4 + 81 + 81 = 6723$. □

Exercise 3 (BMT 2018/4)**.** Find $\sum_{i=1}^{2016} i(i+1)(i+2) \pmod{2018}$. Hint: 1

> **Remark.** The sum $\sum i(i+1)(i+2)$ in Exercise 3 also falls to a weighted difference (discrete integration by parts): with $u_i = i(i+1)(i+2)$, we have $i \cdot u_{i+1} - (i-1) \cdot u_i = 4u_i$, so $\sum_{i=1}^{n} u_i = \frac{n\,u_{n+1}}{4}$ telescopes directly. This is the hockey-stick identity in disguise: the binomial coefficient is hidden inside the weighting.

Exercise 4 (PUMaC 2023 Algebra #B2)**.** The sum $\displaystyle\sum_{m=1}^{2023} \frac{2m}{m^4 + m^2 + 1}$ equals $\frac{a}{b}$ in lowest terms. Find the remainder when $a + b$ is divided by 1000. Hint: 2

Exercise 5 (BMT 2020 Team #9)**.** Define $a_0 = 0$ and $a_n = a_{n-1} + (-1)^n n^2$ for $n \ge 1$. Find a_{100}. Hint: 3

Answers: 0; 105; 5050.

Hints

1. Write $i(i+1)(i+2) = 6\binom{i+2}{3}$; hockey-stick gives $n(n+1)(n+2)(n+3)/4$. At $n = 2016$ a factor of 2018 appears.
2. Factor $m^4 + m^2 + 1 = (m^2 - m + 1)(m^2 + m + 1)$ and check that $\frac{2m}{m^4+m^2+1} = \frac{1}{m^2-m+1} - \frac{1}{m^2+m+1}$; the sum telescopes.
3. Unroll the recurrence into a sum, then pair consecutive terms; each pair is a difference of two squares.