Binomial Sums

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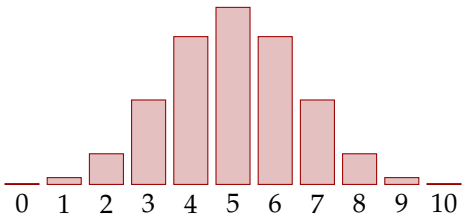
The key idea: two moves — absorb an awkward denominator into the binomial coefficient, or read the sum as one coefficient of a polynomial.



Pascal's row 10: $\binom{10}{k}$

Binomial coefficients are polynomial coefficients in disguise: $\binom{n}{k} = [x^k](1+x)^n$.

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Coefficient Extraction

The coefficient of x^n in $P(x)Q(x)$ is the convolution sum over k.

Coefficient extraction: the setup

Read the summand as one coefficient of a product of polynomials.

When: a sum over k of products of binomial coefficients whose indices add up to a constant. **The move:** recognize each factor as a coefficient, multiply the polynomials, read off one coefficient.

The mechanism: the coefficient of x^n in $P(x)Q(x)$ is the convolution sum $\sum_k [x^k]P \cdot [x^{n-k}]Q$. Since $\binom{n}{k} = \binom{n}{n-k}$ is the coefficient of x^{n-k} in $(1+x)^n$:

$$(1+x)^n \quad \times \quad (1+x)^n \quad \longrightarrow \quad (1+x)^{2n}$$

$$\downarrow$$

$$[x^n] = \binom{2n}{n}$$

Coefficient extraction: Example

Show $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Strategy: write each $\binom{n}{k}$ as a coefficient of $(1+x)^n$ and read the sum as one coefficient of the product.

Coefficient extraction: Solution

$\binom{n}{k} = \binom{n}{n-k}$ is the coefficient of x^{n-k} in $(1+x)^n$. *Why flip to $n-k$: then the two exponents k and $n-k$ add up to the fixed number n.*

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So $\sum_k \binom{n}{k}^2$ is the convolution $\sum_k [x^k](1+x)^n \cdot [x^{n-k}](1+x)^n$ — the coefficient of x^n in $(1+x)^{2n}$:

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}}.$$

Denominator Absorption

Absorb the awkward denominator into the binomial coefficient itself.

Denominator absorption: the setup

Absorb the awkward denominator into the binomial coefficient itself:

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}.$$

When: a $\frac{1}{k+1}$ (or similar) spoils an otherwise binomial-theorem sum. **Why it works:** both sides equal $\frac{n!}{(k+1)!\,(n-k)!}$. After absorbing, the binomial theorem (or the series for e^x) finishes the job.

Denominator absorption: Example

Evaluate $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{k+1}$.

Strategy: absorb the $k+1$ denominator, shift the index, and read off a binomial sum.

Denominator absorption: Solution

Absorb the denominator:

$$\frac{1}{k+1}\binom{n}{k}2^{k+1} = \frac{1}{n+1}\binom{n+1}{k+1}2^{k+1}.$$

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With $j = k+1$, the sum is $\frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j}2^j$. *Why: now it is a full binomial expansion of* $(1+2)^{n+1}$*, missing only the* $j = 0$ *term* 1.

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$$\boxed{\frac{(1+2)^{n+1}-1}{n+1} = \frac{3^{n+1}-1}{n+1}}.$$

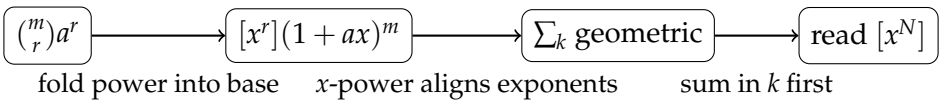
The Hard Variant

When the top index itself moves, build the extracting polynomial directly.

The hard variant: the setup

When the top index itself moves, like $\binom{2018-k}{k-1}$, build the extracting polynomial directly.

When: both indices of $\binom{m}{r}$ change with k, so no single $(1+x)^n$ works. **The move:** give every term its own polynomial, arranged so all terms sit at the *same* power x^N; then add the polynomials first.



$\binom{m}{r}$ is the coefficient of x^r in $(1+x)^m$; a power a^r folds into the base as $(1+ax)^m$, since $[x^r](1+ax)^m = \binom{m}{r}a^r$; and an extra factor x^{N-r} moves that coefficient to x^N.

The hard variant: Example

Compute $\sum_{k=1}^{1009}(-1)^{k+1}\binom{2018-k}{k-1}2^{2019-2k}$.

Strategy: build the extracting polynomial for the term, sum the geometric series in k first, then read off one coefficient. [1]

[1] BMT 2018/8

The hard variant: try it

Compute $\sum_{k=1}^{1009}(-1)^{k+1}\binom{2018-k}{k-1}2^{2019-2k}$.

Hint: the term is (up to the sign) the coefficient of x^{2018} in $x^{2k-1}(2x-1)^{2018-k}$ — check the exponent and the sign. Then sum in k first. [2]

[2] BMT 2018/8

The hard variant: Solution (1/2)

Step 1: one polynomial per term. Since $\binom{2018-k}{k-1} = \binom{2018-k}{2019-2k}$, the term is $\binom{m}{r}a^r$ with $m = 2018-k$, $r = 2019-2k$, $a = 2$; the sign $(-1)^{k-1} = (-1)^{m-r}$ comes free from the base $2x-1$:

$$[x^{2019-2k}](2x-1)^{2018-k} = \binom{2018-k}{k-1} 2^{2019-2k}(-1)^{k-1}.$$

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Step 2: align the exponents. *Why multiply by x^{2k-1}: it moves every term to the same coefficient, x^{2018}.* So each term is $[x^{2018}]\, x^{2k-1}(2x-1)^{2018-k}$.

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Step 3: sum in k first — a geometric series with ratio $\frac{x^2}{2x-1}$ (terms with $k \geq 1010$ contribute nothing to $[x^{2018}]$, since x^{2k-1} already has degree > 2018):

$$\sum_{k\geq 1} x^{2k-1}(2x-1)^{2018-k} = \frac{x(2x-1)^{2017}}{1-\frac{x^2}{2x-1}} = \frac{x(2x-1)^{2018}}{2x-1-x^2}.$$

The hard variant: Solution (2/2)

Step 4: expand around the trouble spot. $2x - 1 - x^2 = -(x-1)^2$. *Why expand* $P(x) = x(2x-1)^{2018}$ *in powers of* $(x-1)$*: only the terms of* P *below* $(x-1)^2$ *leave a fraction.* Since $P(1) = 1$ and $P'(1) = 1 + 2018 \cdot 2 = 4037$,

$$P(x) = 1 + 4037(x-1) + (x-1)^2 R(x), \qquad \text{sum} = -\frac{1}{(x-1)^2} - \frac{4037}{x-1} - R(x).$$

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Step 5: read off $[x^{2018}]$**.** R has degree 2017, so it contributes nothing. With $\frac{1}{(1-x)^2} = \sum (n+1)x^n$ and $\frac{1}{1-x} = \sum x^n$:

$$[x^{2018}]\left(-\frac{1}{(1-x)^2}\right) = -2019, \qquad [x^{2018}]\left(\frac{4037}{1-x}\right) = 4037.$$

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$$\boxed{-2019 + 4037 = 2018}.$$

Practice Exercises

Two problems to try — absorption plus the exponential series, and roots of unity.

Practice: absorption + exponential series

Exercise 1. Let $h_n = \sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{k+1}$. Find $\sum_{n=0}^{\infty} \frac{h_n}{n!}$.

Hint: The worked example gives $h_n = \frac{3^{n+1}-1}{n+1}$. Then recognize the exponential series in $\sum h_n/n!$. [3]

[3] BMT 2018/7

Practice: roots of unity

Exercise 2. Evaluate $\sum_{k=0}^{37} (-1)^k \binom{75}{2k}$.

Hint: $(-1)^k \binom{75}{2k}$ is the real part of $\binom{75}{2k} i^{2k}$; so the sum is the real part of $(1+i)^{75}$. Use $(1+i)^2 = 2i$ to compute it. [4]

[4] BMT 2015 Analysis #7

Answers

1. Exercise 1: $\boxed{e^3 - e}$

2. Exercise 2: $\boxed{-2^{37}}$