Binomial Sums

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Absorb the awkward denominator into the binomial coefficient, or read the sum as one coefficient of a polynomial.

You are given a sum whose terms contain binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ — the number of ways to choose k objects from n — possibly with extra factors like 2^k or a denominator $k+1$, and you need a closed form. Two moves handle almost all of these. The first absorbs an awkward denominator into the binomial coefficient itself via $\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}$, after which the binomial theorem or the series for e^x finishes the job. The second is coefficient extraction: recognize the summand as the coefficient of x^n in some polynomial, turn the sum over k into a geometric series, and read off one coefficient at the end.

Problem 1 (Warm-up)**.** Show $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.

Solution. $\binom{n}{k} = \binom{n}{n-k}$ is the coefficient of x^{n-k} in $(1+x)^n$, so $\sum_k \binom{n}{k}^2$ is the coefficient of x^n in $(1+x)^n(1+x)^n = (1+x)^{2n}$: $\binom{2n}{n}$. □

Problem 2 (Worked example)**.** Evaluate $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{2^{k+1}}{k+1}$.

Solution. Absorb the denominator:

$$\frac{1}{k+1}\binom{n}{k}2^{k+1} = \frac{1}{n+1}\binom{n+1}{k+1}2^{k+1},$$

so with $j = k+1$ the sum is $\frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j}2^j = \frac{3^{n+1}-1}{n+1}$, by the binomial theorem for $(1+2)^{n+1}$ minus its $j=0$ term. Exercise 4 below builds on this. □

When the top index itself moves, like $\binom{2018-k}{k-1}$, no single $(1+x)^n$ works; build the extracting polynomial for each term directly. $\binom{m}{r}$ is the coefficient of x^r in $(1+x)^m$, a power a^r folds into the base as $(1+ax)^m$ (since $[x^r](1+ax)^m = \binom{m}{r}a^r$), and an extra factor x^{N-r} moves every term to the same coefficient x^N. Sum the geometric series in k first, then read off one coefficient.

Problem 3 (BMT 2018/8)**.** Compute $\sum_{k=1}^{1009}(-1)^{k+1}\binom{2018-k}{k-1}2^{2019-2k}$.

Solution. Since $\binom{2018-k}{k-1} = \binom{2018-k}{2019-2k}$, the term is $\binom{m}{r}2^r$ with $m = 2018-k$, $r = 2019-2k$, and the sign $(-1)^{k-1} = (-1)^{m-r}$ comes from the base $2x-1$:

$$[x^{2019-2k}](2x-1)^{2018-k} = \binom{2018-k}{k-1}2^{2019-2k}(-1)^{k-1}.$$

Multiplying by x^{2k-1} moves every term to the same coefficient: the k-th term is $[x^{2018}]\, x^{2k-1}(2x-1)^{2018-k}$. Now sum over k first. Terms with $k \geq 1010$ contribute nothing to $[x^{2018}]$, since x^{2k-1} already has degree > 2018, so we may sum over all $k \geq 1$; it is a geometric series with ratio $\frac{x^2}{2x-1}$:

$$\sum_{k\geq 1} x^{2k-1}(2x-1)^{2018-k} = \frac{x(2x-1)^{2018}}{2x-1-x^2} = -\frac{P(x)}{(x-1)^2}, \qquad P(x) = x(2x-1)^{2018}.$$

Expand P in powers of $x-1$: $P(1) = 1$ and $P'(1) = 1 + 2018 \cdot 2 = 4037$, so $P(x) = 1 + 4037(x-1) + (x-1)^2 R(x)$ with $\deg R = 2017$. Hence

$$\text{sum} = -\frac{1}{(1-x)^2} + \frac{4037}{1-x} - R(x).$$

R contributes nothing to $[x^{2018}]$; from $\frac{1}{(1-x)^2} = \sum (n+1)x^n$ and $\frac{1}{1-x} = \sum x^n$ the answer is $-2019 + 4037 = 2018$. □

Exercise 4 (BMT 2018/7). Let $h_n = \sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{k+1}$. Find $\sum_{n=0}^{\infty} \frac{h_n}{n!}$. Hint: 1

Exercise 5 (BMT 2015 Analysis #7). Evaluate $\displaystyle\sum_{k=0}^{37} (-1)^k \binom{75}{2k}$. Hint: 2

Answers: $e^3 - e$; $\quad -2^{37}$.

Hints

1. The worked example gives $h_n = \frac{3^{n+1}-1}{n+1}$. Then recognize the exponential series in $\sum h_n/n!$.
2. $(-1)^k\binom{75}{2k}$ is the real part of $\binom{75}{2k} i^{2k}$; so the sum is the real part of $(1+i)^{75}$. Use $(1+i)^2 = 2i$ to compute it.