Minimal-Polynomial Reduction

David Liu

The key idea: when a number satisfies a low-degree equation, reduce every power to a linear combination — the coefficients follow a recurrence.

$$\varphi^2 = \varphi + 1 \xrightarrow{\times\varphi} \varphi^3 = 2\varphi + 1 \xrightarrow{\times\varphi} \varphi^4 = 3\varphi + 2 \xrightarrow{\times\varphi} \varphi^5 = 5\varphi + 3$$

Iteration on the exponent, done algebraically: multiply by φ, reduce $\varphi^2 = \varphi + 1$, read off the recurrence.

The Reduction Move

Reduce every power to a linear combination; the coefficients follow a recurrence.

Minimal polynomials: the setup

When a number satisfies a low-degree equation, reduce every power to a linear combination.

When: a high power (or a sum of powers) of a number that satisfies a quadratic, like $\varphi^2 = \varphi + 1$. **The move:** multiply by the number, replace the square using the equation, repeat — the coefficients obey a recurrence.

For the golden ratio φ, the positive root of $x^2 = x + 1$: write $\varphi^n = a_n\varphi + b_n$ with integers a_n, b_n (possible and unique, since φ is irrational). Multiplying by φ and reducing $\varphi^2 = \varphi + 1$ again expresses φ^{n+1} in the same form — forcing a recurrence on the coefficients.

The same move works for any such number: for $\omega = e^{2\pi i/3}$ with $\omega^2 + \omega + 1 = 0$, every power reduces to $a_n\omega + b_n$.

Minimal polynomials: Example

Let φ be the positive root of $x^2 = x + 1$. For $n \geq 0$ let a_n be the unique integer with $\varphi^n - a_n\varphi \in \mathbb{Z}$. Compute $\sum_{n=0}^{10} a_n$.

Strategy: write $\varphi^n = a_n\varphi + b_n$, multiply by φ, reduce, and read off the recurrence on the coefficients. [1]

[1] BMT 2020/4

Minimal polynomials: try it

Let φ be the positive root of $x^2 = x + 1$. For $n \geq 0$ let a_n be the unique integer with $\varphi^n - a_n\varphi \in \mathbb{Z}$. Compute $\sum_{n=0}^{10} a_n$.

Hint: multiply $\varphi^n = a_n\varphi + b_n$ by φ and use $\varphi^2 = \varphi + 1$ to express φ^{n+1} in the same form. What recurrence do a_n and b_n satisfy? [2]

[2] BMT 2020/4

Minimal polynomials: Solution

Write $\varphi^n = a_n\varphi + b_n$ with $b_n \in \mathbb{Z}$ (possible and unique, since φ is irrational). *Why: the problem's a_n is exactly the φ-coefficient.*

Minimal polynomials: Solution

Write $\varphi^n = a_n\varphi + b_n$ with $b_n \in \mathbb{Z}$ (possible and unique, since φ is irrational). *Why: the problem's a_n is exactly the φ-coefficient.*

Multiply by φ and reduce $\varphi^2 = \varphi + 1$ — *why: this steps from n to $n+1$ while staying in the form $a\varphi + b$:*

$$\varphi^{n+1} = a_n\varphi^2 + b_n\varphi = a_n(\varphi + 1) + b_n\varphi = (a_n + b_n)\varphi + a_n.$$

Minimal polynomials: Solution

Write $\varphi^n = a_n\varphi + b_n$ with $b_n \in \mathbb{Z}$ (possible and unique, since φ is irrational). *Why: the problem's a_n is exactly the φ-coefficient.*

Multiply by φ and reduce $\varphi^2 = \varphi + 1$ — *why: this steps from n to $n+1$ while staying in the form $a\varphi + b$:*

$$\varphi^{n+1} = a_n\varphi^2 + b_n\varphi = a_n(\varphi + 1) + b_n\varphi = (a_n + b_n)\varphi + a_n.$$

So $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$ — hence $b_n = a_{n-1}$ and $a_{n+1} = a_n + a_{n-1}$: the Fibonacci recurrence!

Minimal polynomials: Solution

Write $\varphi^n = a_n\varphi + b_n$ with $b_n \in \mathbb{Z}$ (possible and unique, since φ is irrational). *Why: the problem's a_n is exactly the φ-coefficient.*

Multiply by φ and reduce $\varphi^2 = \varphi + 1$ — *why: this steps from n to $n+1$ while staying in the form $a\varphi + b$:*

$$\varphi^{n+1} = a_n\varphi^2 + b_n\varphi = a_n(\varphi + 1) + b_n\varphi = (a_n + b_n)\varphi + a_n.$$

So $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$ — hence $b_n = a_{n-1}$ and $a_{n+1} = a_n + a_{n-1}$: the Fibonacci recurrence! Initial values: $a_0 = 0$ (since $\varphi^0 = 1$) and $a_1 = 1$. Now just list Fibonacci numbers:

Minimal polynomials: Solution

Write $\varphi^n = a_n\varphi + b_n$ with $b_n \in \mathbb{Z}$ (possible and unique, since φ is irrational). *Why: the problem's a_n is exactly the φ-coefficient.*

Multiply by φ and reduce $\varphi^2 = \varphi + 1$ — *why: this steps from n to $n+1$ while staying in the form $a\varphi + b$:*

$$\varphi^{n+1} = a_n\varphi^2 + b_n\varphi = a_n(\varphi + 1) + b_n\varphi = (a_n + b_n)\varphi + a_n.$$

So $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$ — hence $b_n = a_{n-1}$ and $a_{n+1} = a_n + a_{n-1}$: the Fibonacci recurrence! Initial values: $a_0 = 0$ (since $\varphi^0 = 1$) and $a_1 = 1$. Now just list Fibonacci numbers:

$$\boxed{\sum_{n=0}^{10} a_n = 0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143}.$$

Practice Exercises

Two problems to try — reduce, then read off the answer.

Practice: reducing φ^5

Exercise 1. With φ as above, write $\varphi^5 = a\varphi + b$ with integers a, b. Compute $a + b$.

Hint: Iterate $\varphi^{n+1} = (a_n + b_n)\varphi + a_n$ from $a_0 = 0$, $b_0 = 1$ — or read off the Fibonacci numbers.

Practice: reducing x^3

Exercise 2. Suppose a real number x satisfies $x^2 = 10x + 7$. Find the unique ordered pair of integers (m, n) with $x^3 = mx + n$.

Hint: Write $x^3 = x \cdot x^2$, replace x^2 by $10x + 7$, and reduce once more until only x and a constant remain. [3]

[3] CMIMC 2020 A/NT #1

Answers

1. Exercise 1: $\boxed{8}$
2. Exercise 2: $\boxed{(107, 70)}$