Minimal-Polynomial Reduction

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Reduce every power with the minimal polynomial; the coefficients follow a recurrence.

When a number satisfies a low-degree equation — its *minimal polynomial*, the simplest equation it satisfies — reduce every power to a linear combination. For the golden ratio φ, the positive root of $x^2 = x + 1$, this means writing $\varphi^n = a_n\varphi + b_n$ with integers a_n and b_n; multiplying by φ and reducing $\varphi^2 = \varphi + 1$ again expresses φ^{n+1} in the same form, which forces a recurrence on the coefficients — iteration on the exponent, done algebraically. The same move works for any such number: for $\omega = e^{2\pi i/3}$ with $\omega^2 + \omega + 1 = 0$, every power reduces to $a_n\omega + b_n$.

Problem 1 (BMT 2020/4)**.** Let φ be the positive root of $x^2 = x + 1$. For $n \geq 0$ let a_n be the unique integer with $\varphi^n - a_n\varphi \in \mathbb{Z}$. Compute $\sum_{n=0}^{10} a_n$.

Solution. Write $\varphi^n = a_n\varphi + b_n$ with $b_n \in \mathbb{Z}$ (possible and unique since φ is irrational). Multiply by φ and use $\varphi^2 = \varphi + 1$:

$$\varphi^{n+1} = a_n\varphi^2 + b_n\varphi = a_n(\varphi + 1) + b_n\varphi = (a_n + b_n)\varphi + a_n,$$

so $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$. Hence $b_n = a_{n-1}$ and $a_{n+1} = a_n + a_{n-1}$: the Fibonacci recurrence. With $a_0 = 0$ (since $\varphi^0 = 1$) and $a_1 = 1$,

$$\sum_{n=0}^{10} a_n = 0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143.$$

□

Exercise 2. With φ as above, write $\varphi^5 = a\varphi + b$ with integers a, b. Compute $a + b$. Hint: 1

Exercise 3 (CMIMC 2020 A/NT #1)**.** Suppose a real number x satisfies $x^2 = 10x + 7$. Find the unique ordered pair of integers (m, n) with $x^3 = mx + n$. Hint: 2

Answers: 8; $(107, 70)$.

Hints

1. Iterate $\varphi^{n+1} = (a_n + b_n)\varphi + a_n$ from $a_0 = 0$, $b_0 = 1$ — or read off the Fibonacci numbers.
2. Write $x^3 = x \cdot x^2$, replace x^2 by $10x + 7$, and reduce once more until only x and a constant remain.