Vieta's Dictionary

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The key idea: symmetric equations in x, y, z *are* Vieta's e_1, e_2, e_3 in disguise — read them as the polynomial $t^3 - e_1 t^2 + e_2 t - e_3$ and work with that instead.

$$\begin{matrix} x+y+z=6 \\ xy+yz+zx=11 \\ xyz=6 \end{matrix} \xrightarrow{\text{read}} \begin{matrix} e_1=6 \\ e_2=11 \\ e_3=6 \end{matrix} \xrightarrow{\text{build}} \begin{matrix} t^3-6t^2+11t-6 \\ =(t-1)(t-2)(t-3) \end{matrix}$$

The equations hand you the polynomial; the polynomial hands you the roots.

The Dictionary

When a problem hands you symmetric equations, read them as Vieta's data and build the polynomial.

The technique in two sentences

When to reach for it: the problem is symmetric in its unknowns — it only involves sums like $x + y + z$, $xy + yz + zx$, xyz — or it asks about the roots of a polynomial you cannot solve.

The move: translate between the two languages. Symmetric data about the numbers $\longleftrightarrow$ coefficients of the polynomial whose roots they are. Build that polynomial (or read its coefficients), and let it do the work.

Vieta's dictionary: the setup

Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ have roots $r_1, \ldots, r_n$. Then

$$\begin{aligned} e_1 &= r_1 + \cdots + r_n = -a_{n-1}, \\ e_2 &= \sum_{i<j} r_i r_j = a_{n-2}, \\ &\vdots \\ e_k &= \sum_{i_1<\cdots<i_k} r_{i_1} \cdots r_{i_k} = (-1)^k a_{n-k}. \end{aligned}$$

The case you will use most is the cubic $x^3 + ax^2 + bx + c$ with roots r_1, r_2, r_3:

$$\sum r_i = -a, \qquad \sum_{i<j} r_i r_j = b, \qquad r_1 r_2 r_3 = -c.$$

Watch the signs: they alternate. If you ever doubt a sign, expand $(x - r_1)(x - r_2)(x - r_3)$ and compare — ten seconds, zero risk.

Warm-up Example

Symmetric equations in x, y, z are just a cubic waiting to be built.

Vieta's dictionary: Example (statement)

The real numbers x, y, z satisfy

$$x + y + z = 6, \qquad xy + yz + zx = 11, \qquad xyz = 6.$$

Find x, y, z.

Strategy: read the equations as Vieta's e_1, e_2, e_3 and build the cubic.

Vieta's dictionary: Example (solution)

The equations say $e_1 = 6, e_2 = 11, e_3 = 6$, so x, y, z are the three roots of

$$t^3 - 6t^2 + 11t - 6 = 0.$$

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$$\boxed{\{x, y, z\} = \{1, 2, 3\}}.$$

Main Example

A problem that looks analytic but surrenders to the dictionary.

Vieta's dictionary: Main example (statement)

Count the positive integer triples (B, M, T) with $B, M, T < 24$ satisfying

$$BM + MT + BT = (B + M + T)\sqrt[3]{BMT}.$$

Strategy: the equation is symmetric — force a cube root out of it, then factor in the new variables. [1]

[1] BMT 2024/7

Vieta's dictionary: Main example (solution, part 1)

The left side and $B + M + T$ are integers, so $\sqrt[3]{BMT} = \frac{BM+MT+BT}{B+M+T}$ is rational, hence a positive integer k; then $BMT = k^3$ and

$$BM + BT + MT = k(B + M + T). \tag{1}$$

Why first: a cube root sitting among integers is a red flag — force it to be an integer before anything else. (A rational cube root of an integer is an integer.)

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Why scale: (1) is homogeneous, so dividing by k makes the product 1. Put $X = B/k$, $Y = M/k$, $Z = T/k$. Then

$$XYZ = 1, \qquad XY + YZ + ZX = X + Y + Z. \tag{2}$$

Vieta's dictionary: Main example (solution, part 2)

With $Z = 1/(XY)$, multiply (2) by XY:

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$XY = 1$ means $Z = 1$, so one of X, Y, Z is 1; say $X = 1$. Then $B = k$ and $MT = k^2 = B^2$, i.e.

$$MT = B^2. \tag{3}$$

Vieta's dictionary: Main example (solution, part 3)

Divide by $g = \gcd(B, M, T)$. Then M, T are coprime (a common prime would divide B) with square product, so both are squares:

$$(B, M, T) = (rs,\ r^2,\ s^2) \text{ up to order, with } \gcd(r, s) = 1,$$

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and scaling back by g gives all solutions. Now count with entries < 24:

- **Constant:** (d, d, d), $d = 1, \ldots, 23$ — 23 triples.
- **Non-constant primitive:** $r^2, s^2 < 24$ gives $r, s \leq 4$; with $\gcd(r, s) = 1$ and $\{r, s\} \neq \{1, 1\}$: $(1, 2, 4)$, $(1, 3, 9)$, $(1, 4, 16)$, $(4, 6, 9)$, $(9, 12, 16)$ up to order.

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Each type scales by d staying < 24: $5 \cdot 6 + 2 \cdot 6 + 1 \cdot 6 + 2 \cdot 6 + 1 \cdot 6 = 66$ triples (6 orderings each).

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$$\boxed{23 + 66 = 89}.$$

Alternative: The Vieta Move

To test whether a suspected value is one of the unknowns, evaluate the Vieta cubic at it.

The Vieta move: guess-and-test

Let $p = B + M + T$, $q = BM + BT + MT$, $r = BMT$. Then B, M, T are the roots of

$$x^3 - px^2 + qx - r.$$

The reusable move: to test whether a suspected value v is one of the unknowns, evaluate this cubic at v — a zero means v is one of them.

Here, from (1), $q = kp$ and $r = k^3$. Test $v = q/p$:

$$-pv^2 + qv = v(q - pv) = 0, \qquad v^3 - r = (q/p)^3 - k^3 = k^3 - k^3 = 0.$$

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$$-pv^2 + qv = v(q - pv) = 0, \qquad v^3 - r = (q/p)^3 - k^3 = k^3 - k^3 = 0.$$

So q/p is a root — say $B = q/p$. Then $q = pB$ gives $MT = B^2$, and we finish as before.

Practice Exercises

Two problems to try — the same moves you just saw.

Practice: Vieta's dictionary

Exercise 1. Let a, b, c be the roots of $x^3 - 5x^2 + 6x - 1$. Compute $a^2 + b^2 + c^2$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Hint: $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$; $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$.

Practice: Vieta's dictionary (continued)

Exercise 2. Nonzero complex numbers x, y, z satisfy $(x+y+z)(x^2+y^2+z^2) = x^3+y^3+z^3$. Compute

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Hint: read the equation in the dictionary entries e_1, e_2, e_3: $x^2+y^2+z^2 = e_1^2 - 2e_2$ and $x^3+y^3+z^3 = e_1^3 - 3e_1e_2 + 3e_3$. The target is $\frac{e_1e_2}{e_3}$. [2]

[2] CMIMC 2017 Team #2

Answers

1. Exercise 1: $\boxed{13}$ and $\boxed{6}$
2. Exercise 2: $\boxed{3}$