Vieta's Formulas

David Liu

Symmetric equations in x, y, z are Vieta's e_1, e_2, e_3 in disguise: read them as the polynomial $t^3 - e_1 t^2 + e_2 t - e_3$ and work with that instead.

Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ have roots $r_1, \ldots, r_n$ (possibly complex, possibly repeated). Then

$$\begin{aligned} e_1 &= r_1 + \cdots + r_n = -a_{n-1}, \\ e_2 &= \sum_{i<j} r_i r_j = a_{n-2}, \\ &\vdots \\ e_k &= \sum_{i_1 < \cdots < i_k} r_{i_1} \cdots r_{i_k} = (-1)^k a_{n-k}. \end{aligned}$$

The case you will use most is the cubic $x^3 + ax^2 + bx + c$ with roots r_1, r_2, r_3:

$$\sum r_i = -a, \qquad \sum_{i<j} r_i r_j = b, \qquad r_1 r_2 r_3 = -c.$$

Watch the signs: they alternate. If you ever doubt a sign, expand $(x - r_1)(x - r_2)(x - r_3)$ and compare — ten seconds, zero risk.

Problem 1 (Warm-up)**.** The real numbers x, y, z satisfy $x + y + z = 6$, $xy + yz + zx = 11$, $xyz = 6$. Find x, y, z.

Solution. They are the three roots of $t^3 - 6t^2 + 11t - 6$. Since $t = 1$ is a root, $t^3 - 6t^2 + 11t - 6 = (t-1)(t-2)(t-3)$, so $\{x, y, z\} = \{1, 2, 3\}$. □

Problem 2 (BMT 2024/7)**.** Count the positive integer triples (B, M, T) with $B, M, T < 24$ satisfying $BM + MT + BT = (B + M + T)\sqrt[3]{BMT}$.

Solution. The left side and $B + M + T$ are integers, so $\sqrt[3]{BMT} = \frac{BM+MT+BT}{B+M+T}$ is rational. A rational cube root of a positive integer is a positive integer k (rational root theorem on $x^3 - BMT$), so $BMT = k^3$ and

$$BM + BT + MT = k(B + M + T). \tag{1}$$

The equation is homogeneous, so scale the product to 1: put $X = B/k$, $Y = M/k$, $Z = T/k$. Then

$$XYZ = 1, \qquad XY + YZ + ZX = X + Y + Z. \tag{2}$$

Substituting $Z = 1/(XY)$ into (2) and multiplying by XY gives $X^2Y^2 - X^2Y - XY^2 + X + Y - 1 = 0$, which factors as

$$(X-1)(Y-1)(XY-1) = 0.$$

Since $XY = 1$ means $Z = 1$, one of X, Y, Z is 1; by symmetry say $X = 1$. Then $B = k$ and $MT = k^3/B = k^2$, i.e.

$$MT = B^2. \tag{3}$$

Now let $g = \gcd(B, M, T)$ and divide it out. In a solution of (3) with $\gcd(B, M, T) = 1$, a prime dividing both M and T would divide B^2, hence B; so M and T are coprime with square product, hence both squares: $(B, M, T) = (rs, r^2, s^2)$ with $\gcd(r, s) = 1$. Up to order, every solution is $g \cdot (rs, r^2, s^2)$ — a geometric progression. Count entries < 24:

- **Constant** ($r = s = 1$): (d, d, d) for $d = 1, \ldots, 23$ — 23 triples.
- **Non-constant:** $r^2, s^2 < 24$ forces $r, s \leq 4$; the coprime pairs with $r \neq s$ give $\{1, 2, 4\}$, $\{1, 3, 9\}$, $\{1, 4, 16\}$, $\{4, 6, 9\}$, $\{9, 12, 16\}$. Scaling by g keeps entries below 24 for $5, 2, 1, 2, 1$ values of g, and each has 6 orderings: $6(5 + 2 + 1 + 2 + 1) = 66$ triples.

The answer is $23 + 66 = 89$. □

> **Remark.** The Vieta move gets to (3) faster. With $p = B + M + T$, $q = BM + BT + MT$, $r = BMT$, the numbers B, M, T are the roots of $x^3 - px^2 + qx - r$. To test whether a suspected value v is one of the unknowns, evaluate this cubic at v. By (1), $q = kp$ and $r = k^3$; at $v = q/p = k$ the middle terms vanish ($-pv^2 + qv = v(q - pv) = 0$) and $v^3 - r = k^3 - k^3 = 0$. So q/p is a root — say B. Then $q = pB$ gives $MT = B^2$.

Exercise 3. Let a, b, c be the roots of $x^3 - 5x^2 + 6x - 1$. Compute $a^2 + b^2 + c^2$ and $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$. Hint: 1

Exercise 4 (CMIMC 2017 Team #2)**.** Nonzero complex numbers x, y, z satisfy $(x + y + z)(x^2 + y^2 + z^2) = x^3 + y^3 + z^3$. Compute $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$. Hint: 2

Answers: 13, 6; 3.

Hints

1. $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$; $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$.
2. Read the equation in the dictionary entries e_1, e_2, e_3: $x^2 + y^2 + z^2 = e_1^2 - 2e_2$ and $x^3 + y^3 + z^3 = e_1^3 - 3e_1e_2 + 3e_3$. The target is $\frac{e_1e_2}{e_3}$.