Transforming Roots

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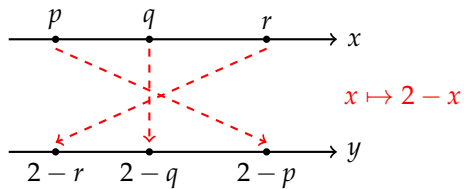
The key idea: if the target is symmetric in the new roots, build it straight from Vieta — no solving for anything. In the example, each new root is $2 -$ (an old root): an old root *mirrored* through 1.



$s_1 = p + q = 2 - r$, $s_2 = p + r = 2 - q$, $s_3 = q + r = 2 - p$: the mirror reverses the order.

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The Two Viewpoints

Given the roots of one polynomial, build the polynomial whose roots are made from them — two viewpoints, one selection rule.

The technique in two sentences

When to reach for it: you know a polynomial (so, via Vieta, the symmetric functions of its roots p, q, r), and the question is about a polynomial whose roots are new numbers built from p, q, r.

The move: never solve for p, q, r. Either compute the new roots' sum, pairwise products and product directly from Vieta (they come out symmetric in p, q, r), or substitute to get the equation each new root satisfies.

Recall Vieta: for $x^3 + ax^2 + bx + c$ with roots p, q, r: $p + q + r = -a$, $pq + pr + qr = b$, $pqr = -c$.

Root transformations: the setup

Sometimes you must construct a polynomial from an existing one: given the roots p, q, r of a known polynomial, find the polynomial whose roots are new numbers built from them, like $p+q$, $p+r$, $q+r$.

Two viewpoints:

- **Symmetric computation:** if the target is symmetric in the new numbers, compute its elementary symmetric polynomials *directly from the old roots* — there is no need to solve for anything.
- **Substitution:** find the equation each new number satisfies, then read off its coefficients.

Selection rule: symmetric target $\rightarrow$ symmetric computation; each new root a function of a single old root $\rightarrow$ substitution. When both hold (as in today's example), either works.

Example

New roots $p+q$, $p+r$, $q+r$ are symmetric in p, q, r — so the symmetric computation applies (and, as we will see, substitution does too).

Root transformations: Example (statement)

Let p, q, r be the three roots of $x^3 - 2x^2 + 3x - 2023$. Suppose $x^3 + Bx^2 + Mx + T$ has roots $p + q$, $p + r$, $q + r$. Compute $B - M + T$.

Strategy: the new roots are symmetric in p, q, r — compute their e_1, e_2, e_3 straight from Vieta. [1]

[1] BMT 2023/5

Root transformations: Solution (step 1: B)

Vieta on $x^3 - 2x^2 + 3x - 2023$ gives

$$p + q + r = 2, \qquad pq + pr + qr = 3, \qquad pqr = 2023.$$

Root transformations: Solution (step 1: B)

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Write $s_1 = p + q$, $s_2 = p + r$, $s_3 = q + r$. As a package these are symmetric in p, q, r. *Why that matters:* swapping p, q, r just permutes the s_i, so every symmetric function of the s_i is symmetric in p, q, r — and Vieta gives those.

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By Vieta on the new cubic, B is minus the sum of its roots, and each old root appears in two of the s_i:

$$B = -(s_1 + s_2 + s_3) = -2(p + q + r) = -4.$$

Root transformations: Solution (step 2: M)

For M, note $(p+q)(p+r) = p^2 + p(q+r) + qr$; summing cyclically gives each square once and each product pq three times. Then rewrite $p^2 + q^2 + r^2 = (p+q+r)^2 - 2(pq+pr+qr)$ — *why:* sums of squares are not Vieta data, but this identity turns them into Vieta data.

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$$\begin{aligned} M &= s_1s_2 + s_1s_3 + s_2s_3 \\ &= (p^2+q^2+r^2) + 3(pq+pr+qr) \\ &= \big((p+q+r)^2 - 2(pq+pr+qr)\big) + 3(pq+pr+qr) \\ &= (4-6)+9 = 7. \end{aligned}$$

Root transformations: Solution (step 3: T)

Since $p+q+r=2$, each new root is 2 minus the missing old root: $s_1 = 2-r$, $s_2 = 2-q$, $s_3 = 2-p$. *Why:* a product of sums is messy; a product of $(2-\text{root})$ is just the cubic evaluated at 2.

$$\begin{aligned} s_1 s_2 s_3 &= (2-p)(2-q)(2-r) \\ &= 8 - 4(p+q+r) + 2(pq+pr+qr) - pqr \\ &= 8 - 8 + 6 - 2023 = -2017, \end{aligned}$$

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$$\boxed{B - M + T = -4 - 7 + 2017 = 2006}.$$

The Substitution Viewpoint

Each new root is a function of a single old root — find the equation the new root satisfies.

The substitution viewpoint

A second way to see the example. Each new root is $2 -$ (an old root): $s_1 = p + q = 2 - r$, and so on. Put $y = 2 - x$: every s_i satisfies

$$(2 - y)^3 - 2(2 - y)^2 + 3(2 - y) - 2023 = 0,$$

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i.e. $y^3 - 4y^2 + 7y + 2017 = 0$ — the same B, M, T as before. Here *both* viewpoints apply: the new roots are a symmetric package, and each is a function of a single old root ($x \mapsto 2 - x$). Either works. In general: symmetric target $\to$ symmetric computation; new roots f(one old root) $\to$ substitution.

Practice Exercises

Two problems to try — one symmetric, one a root map.

Practice: transforming roots

Exercise 1. Let p, q, r be the roots of $x^3 - 3x^2 + x + 2$. Compute $(p+q)(q+r)(r+p)$.

Hint: write $p + q = 3 - r$ and expand $(3-p)(3-q)(3-r)$ via Vieta.

Exercise 2. Let $f(x) = (2x^2 + 7x + 3)(x^2 - 2x - 15)$. Translate the graph 4 units to the right, then rotate 180° about the origin, obtaining g. Compute the sum of the *distinct* roots of g.

Hint: factor f first. A right shift by 4 followed by a 180° rotation about the origin sends each root r to $-r - 4$. The repeated root of f must not be double-counted. [2]

[2] BMT 2025/4

Answers

1. Exercise 1: $\boxed{5}$
2. Exercise 2: $\boxed{-\frac{27}{2}}$