Transforming Roots

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New roots built from old ones? If the new roots form a symmetric package, compute their e_1, e_2, e_3 straight from Vieta — no solving for anything.

Sometimes you need to construct a polynomial from an existing one: given the roots p, q, r of a known polynomial, find the polynomial whose roots are new numbers built from them, like $p+q$, $p+r$, $q+r$. The core move: if the target is symmetric in the new numbers, compute its elementary symmetric polynomials directly from the old roots — there is no need to solve for anything. A second viewpoint: substitute to find the equation the new numbers satisfy, then read off its coefficients.

Recall Vieta: for x^3+ax^2+bx+c with roots p, q, r: $p+q+r=-a$, $pq+pr+qr=b$, $pqr=-c$.

Problem 1 (BMT 2023/5)**.** Let p, q, r be the three roots of $x^3-2x^2+3x-2023$. Suppose x^3+Bx^2+Mx+T has roots $p+q$, $p+r$, $q+r$. Compute $B-M+T$.

Solution. Vieta on $x^3-2x^2+3x-2023$ gives

$$p+q+r=2, \qquad pq+pr+qr=3, \qquad pqr=2023.$$

Write $s_1=p+q$, $s_2=p+r$, $s_3=q+r$. As a package these are symmetric in p, q, r, so their elementary symmetric polynomials come straight from Vieta:

$$\begin{aligned} B &= -(s_1+s_2+s_3) = -2(p+q+r) = -4, \\ M &= s_1s_2+s_1s_3+s_2s_3. \end{aligned}$$

For M, note $(p+q)(p+r)=p^2+p(q+r)+qr$; summing cyclically,

$$\begin{aligned} M &= (p^2+q^2+r^2)+3(pq+pr+qr) \\ &= \left[(p+q+r)^2-2(pq+pr+qr)\right]+3(pq+pr+qr) = (4-6)+9=7. \end{aligned}$$

Finally, since $p+q+r=2$, each new root is 2 minus the missing old root: $s_1=2-r$, $s_2=2-q$, $s_3=2-p$. So

$$\begin{aligned} s_1s_2s_3=(2-p)(2-q)(2-r) &= 8-4(p+q+r)+2(pq+pr+qr)-pqr \\ &= 8-8+6-2023=-2017, \end{aligned}$$

and $T=-s_1s_2s_3=2017$. Hence $B-M+T=-4-7+2017=2006$. □

Remark. A second way, the substitution viewpoint: each new root is $2-$ (an old root), e.g. $s_1 = p+q = 2-r$. So with $y = 2-x$, each s_i satisfies $(2-y)^3 - 2(2-y)^2 + 3(2-y) - 2023 = 0$, i.e. $y^3 - 4y^2 + 7y + 2017 = 0$. Same B, M, T. Here both viewpoints apply — the new roots are a symmetric package, *and* each one is a function of a single old root — so either works. In general: a symmetric target calls for the symmetric computation; new roots of the form f(one old root) call for substitution.

Exercise 2. Let p, q, r be the roots of $x^3 - 3x^2 + x + 2$. Compute $(p+q)(q+r)(r+p)$. Hint: 1

Exercise 3 (BMT 2025/4)**.** Let $f(x) = (2x^2 + 7x + 3)(x^2 - 2x - 15)$. Translate the graph 4 units to the right, then rotate 180° about the origin, obtaining g. Compute the sum of the *distinct* roots of g. Hint: 2

Answers: 5; $-\dfrac{27}{2}$.

Hints

1. Write $p + q = 3 - r$ and expand $(3-p)(3-q)(3-r)$ via Vieta.
2. Factor f first. A right shift by 4 followed by a 180° rotation about the origin sends each root r to $-r-4$. The repeated root of f must not be double-counted.