Power Sums: Multiply and Add

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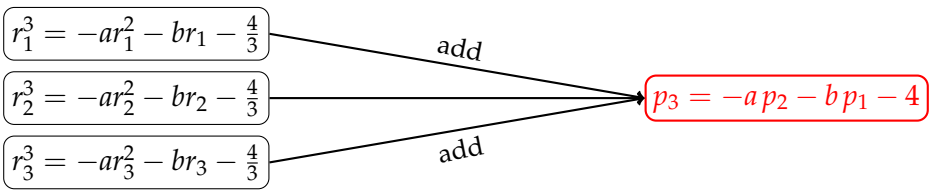
The key idea: every root satisfies its own polynomial equation — multiply that equation by a power of the root, add the results over all roots, and a recurrence for the power sums falls out. Nothing to memorize.



Three copies of the root equation, added up, become one recurrence for $p_k = r_1^k + r_2^k + r_3^k$.

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The Move: Multiply and Add

Given a polynomial and asked for a sum of powers of its roots: multiply each root's equation by a power, add them up.

The technique in two sentences

When to reach for it: you know a polynomial but not its roots, and you are asked for $r_1^k + r_2^k + \cdots$ (or a condition on it).

The move: write the equation each root satisfies, multiply it by r^{k-n} (where n is the degree), and add over all roots. Every term becomes a power sum, so p_k is expressed through $p_{k-1}, \ldots, p_{k-n}$ — climb from the start values.

Example: roots of $x^2 = x + 1$ give $r^k = r^{k-1} + r^{k-2}$, so $p_k = p_{k-1} + p_{k-2}$.

Power sums: the setup

You are given a polynomial with roots r_1, r_2, r_3 and asked for a *power sum*

$$p_k = r_1^k + r_2^k + r_3^k.$$

The classical formulas relating these to the coefficients (Newton's identities) are easy to misremember, so we use a sturdier move.

The move: every root satisfies its own polynomial equation. Multiply that equation by a power of the root and add the results over all roots — you get a recurrence computing p_k from earlier power sums, starting from p_1 and p_2 (straight from the coefficients). No memorization.

Example

A sum of cubes that looks impossible without knowing the roots — the recurrence finds it from the coefficients alone.

Power sums: Example (statement)

Let r_1, r_2, r_3 be the (possibly complex) roots of $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers (a, b) satisfy

$$r_1^3 + r_2^3 + r_3^3 = 0?$$

Strategy: build the power-sum recurrence from the polynomial itself; the condition becomes a divisibility chase on a and b. [1]

[1] BMT 2019/7

Power sums: Solution (step 1: the recurrence)

Start values. $p_1 = \sum r_i = -a$ and $p_2 = \sum r_i^2 = p_1^2 - 2\sum r_i r_j = a^2 - 2b$.
Why: these come straight from Vieta; the recurrence needs them to get going.

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Multiply and add. Each root satisfies $r^3 = -ar^2 - br - \frac{4}{3}$ (already degree 3, so multiply by r^0); summing over the three roots:

$$p_3 = -ap_2 - bp_1 - 3 \cdot \tfrac{4}{3} = -a(a^2 - 2b) + ab - 4 = -a^3 + 3ab - 4.$$

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Use the condition. $p_3 = 0$ gives an equation in integers:

$$a^3 - 3ab + 4 = 0, \qquad \text{i.e.} \qquad a(a^2 - 3b) = -4.$$

Power sums: Solution (step 2: the divisibility chase)

Why this form: a product of integers equal to -4 leaves few options. From $a(a^2 - 3b) = -4$: $a \mid 4$, so $a \in \{\pm 1, \pm 2, \pm 4\}$, and $b = \frac{1}{3}\left(\frac{4}{a} + a^2\right)$ must be an integer.

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a	$\frac{4}{a} + a^2$	b
1	5	—
-1	-3	-1
2	6	2
-2	2	—
4	17	—
-4	15	5

Power sums: Solution (step 2: the divisibility chase)

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4	17	—
-4	15	5

Three pairs: $(2, 2)$, $(-1, -1)$, $(-4, 5)$.

Practice Exercises

Two problems to try — the same multiply-and-add move.

Practice: power sums

Exercise 1. Let a, b, c be real numbers with $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $abc = 5$. Write

$$\left(a - \frac{1}{b}\right)^3 + \left(b - \frac{1}{c}\right)^3 + \left(c - \frac{1}{a}\right)^3 = \frac{m}{n}$$

in lowest terms, and compute $m + n$.

Hint: set $x = a - \frac{1}{b}$, $y = b - \frac{1}{c}$, $z = c - \frac{1}{a}$. What is $x + y + z$? Then $x^3 + y^3 + z^3 = 3xyz$; expand xyz and use the given conditions. [2]

[2] BMT 2020/7

Practice: power sums (continued)

Exercise 2. Let a and b be the roots of $x^2 - 2x + 5$. Compute $|a^8 + b^8|$.

Hint: each root satisfies $r^2 = 2r - 5$. Multiply by r^{k-2} and add over both roots: $p_k = 2p_{k-1} - 5p_{k-2}$ for $p_k = a^k + b^k$. Build up from $p_0 = 2$ and $p_1 = 2$. [3]

[3] PUMaC 2011 Algebra #B2

Answers

1. Exercise 1: $\boxed{77}$
2. Exercise 2: $\boxed{1054}$