Power Sums: Multiply and Add

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Asked for a sum of powers of the roots? Multiply each root's own equation by a power of the root and add over all roots: a recurrence for the power sums falls out, with nothing to memorize.

You are given a polynomial with roots r_1, r_2, r_3 and asked for a *power sum* like $r_1^3 + r_2^3 + r_3^3$ — the sum of the kth powers of the roots, written $p_k = r_1^k + r_2^k + r_3^k$. The classical formulas relating these to the coefficients (Newton's identities) are easy to misremember, so we use a sturdier move: every root satisfies its own polynomial equation, so multiply that equation by a power of the root and add the results over all roots to get a recurrence that computes p_k from earlier power sums. Starting from p_1 and p_2, which come straight from the coefficients, this reaches any p_k with no memorization.

Problem 1 (BMT 2019/7)**.** Let r_1, r_2, r_3 be the (possibly complex) roots of $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers (a, b) satisfy $r_1^3 + r_2^3 + r_3^3 = 0$?

Solution. Put $p_1 = \sum r_i = -a$ and $p_2 = \sum r_i^2 = p_1^2 - 2\sum r_i r_j = a^2 - 2b$. Each root satisfies $r^3 = -ar^2 - br - \frac{4}{3}$; summing over the three roots,

$$p_3 = -ap_2 - bp_1 - 3 \cdot \tfrac{4}{3} = -a(a^2 - 2b) + ab - 4 = -a^3 + 3ab - 4.$$

The condition $p_3 = 0$ gives $a^3 - 3ab + 4 = 0$, i.e. $a(a^2 - 3b) = -4$. Hence $a \mid 4$, so $a \in \{\pm 1, \pm 2, \pm 4\}$, and $b = \frac{1}{3}\left(\frac{4}{a} + a^2\right)$ must be an integer:

a	$\frac{4}{a} + a^2$	b
1	5	—
-1	-3	-1
2	6	2
-2	2	—
4	17	—
-4	15	5

Three pairs: $(2, 2)$, $(-1, -1)$, $(-4, 5)$. □

> **Remark.** A second way: the official solution uses $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ — that identity *is* the multiply-and-add above, packaged. The unpackaged version never needs to be remembered.

Exercise 2 (BMT 2020/7)**.** Let a, b, c be real numbers with $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $abc = 5$. Write $\left(a - \frac{1}{b}\right)^3 + \left(b - \frac{1}{c}\right)^3 + \left(c - \frac{1}{a}\right)^3 = \frac{m}{n}$ in lowest terms, and compute $m + n$. Hint: 1

Exercise 3 (PUMaC 2011 Algebra #B2)**.** Let a and b be the roots of $x^2 - 2x + 5$. Compute $|a^8 + b^8|$. Hint: 2

Answers: 77; 1054.

Hints

1. Set $x = a - \frac{1}{b}$, $y = b - \frac{1}{c}$, $z = c - \frac{1}{a}$. What is $x + y + z$? Then $x^3 + y^3 + z^3 = 3xyz$; expand xyz and use the given conditions.
2. Each root satisfies $r^2 = 2r - 5$. Multiply by r^{k-2} and add over both roots: $p_k = 2p_{k-1} - 5p_{k-2}$ for $p_k = a^k + b^k$. Build up from $p_0 = 2$ and $p_1 = 2$.