Symmetric Rational Sums

David Liu

The key idea: the logarithmic derivative $\dfrac{P'(x)}{P(x)} = \sum_i \dfrac{1}{x - r_i}$ collapses a whole symmetric sum of fractions into *one* evaluation of P and P'.

$$\sum_{i=1}^{2020} \frac{1}{1 - z_i} \xrightarrow{\text{one evaluation}} \frac{P'(1)}{P(1)} = \frac{2020 \cdot 2021/2}{2021}$$

$$\boxed{1010}$$

Two thousand fractions in, one fraction out.

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The Logarithmic Derivative

A symmetric sum of fractions looks hopeless to clear by hand — the logarithmic derivative collapses it.

The technique in two sentences

When to reach for it: you must add the same fraction over all roots of a polynomial, $\sum_i \frac{1}{c-r_i}$ or something you can massage into that shape.

The move: that sum is exactly $\dfrac{P'(c)}{P(c)}$ — evaluate P and P' at one point and divide. If the fraction has the wrong shape, split it (partial fractions) or rescale it until each piece looks like $\frac{1}{c-r_i}$.

Rational sums: the setup

You are given a polynomial P with roots $r_1, \ldots, r_n$ and asked for a *symmetric sum of rational expressions* — you plug each root into the same fraction, like $\frac{1}{1-r}$, and add the results:

$$\sum_{i=1}^{n} \frac{1}{1-r_i}.$$

Clearing all those denominators by hand is hopeless when n is large.

The core move: the logarithmic derivative

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x-r_i},$$

which collapses the entire sum into one evaluation of P and its derivative.

(Why: for $P(x) = \prod_i (x-r_i)$ the product rule gives $P'(x) = \sum_i \prod_{j \neq i} (x-r_j)$; divide by $P(x)$.)

Example

Two thousand roots of unity, one sum — watch the derivative do all the work.

Rational sums: Example (statement)

Let $z_1, \ldots, z_{2020}$ be the roots of

$$z^{2020} + z^{2019} + \cdots + z + 1.$$

Compute $\displaystyle\sum_{i=1}^{2020} \frac{1}{1 - z_i}$.

Strategy: recognize the sum as $\frac{P'(1)}{P(1)}$ for this P — then it is just arithmetic. [1]

[1] BMT 2021/7

Rational sums: Example (solution)

With $P(z) = z^{2020} + \cdots + 1$, the sum is exactly $P'(1)/P(1)$.
Why: each term $\frac{1}{1-z_i}$ is $\frac{1}{x-r_i}$ at $x = 1$ — the shape of the move.

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Now $P(1) = 2021$ (2021 terms, each 1) and, differentiating term by term,

$$P'(1) = 2020 + 2019 + \cdots + 1 = \frac{2020 \cdot 2021}{2},$$

so the sum is

$$\frac{2020 \cdot 2021/2}{2021} = \boxed{1010}.$$

The Split Trick

When the denominator doesn't come from a single shift, split it first — then apply the move twice.

Rational sums: splitting

When the denominator doesn't come from a single shift, split it first:

$$\frac{1}{1-r^2} = \tfrac{1}{2}\left(\frac{1}{1-r} + \frac{1}{1+r}\right).$$

Why: $\frac{1}{1-r^2}$ is not of the form $\frac{1}{c-r}$, but each piece is (or nearly is) — so the move applies to each piece separately.

For $\sum \frac{1}{1-r_i}$ use $P'(1)/P(1)$. For $\sum \frac{1}{1+r_i}$, note $\frac{1}{1+r_i} = \frac{1}{1-(-r_i)}$: apply the move to $Q(x) = P(-x)$, whose roots are the negatives of the roots of P. (Or apply Vieta to the shifted polynomials $P(1-x)$ and $P(x-1)$.)

Rational sums: try the split

Exercise 1. Let r, s, t be the distinct roots of $x^3 - 2022x^2 + 2022x + 2022$. Compute

$$\frac{1}{1-r^2} + \frac{1}{1-s^2} + \frac{1}{1-t^2}.$$

Hint: split $\frac{1}{1-r^2} = \frac{1}{2}\left(\frac{1}{1-r} + \frac{1}{1+r}\right)$. The first sum is $P'(1)/P(1)$ as in the example; for the second, use the same move on $Q(x) = P(-x)$. [2]

[2] BMT 2022/7

Practice Exercises

A roots-of-unity sum that the logarithmic derivative handles in two lines.

Practice: rational sums

Exercise 2. Let $z = \frac{\sqrt{2}+i\sqrt{2}}{2}$. Given that

$$\sum_{k=0}^{13} \frac{1}{1-ze^{ik\pi/7}} = a - bi$$

with real numbers a and b, find $a + b$.

Hint: the 14 numbers $e^{ik\pi/7}$ are the roots of $P(t) = t^{14} - 1$. Write $\frac{1}{1-z\omega_k} = \frac{1/z}{1/z-\omega_k}$ and apply the logarithmic derivative to P at $x = 1/z$; then compute z^{14}. [3]

[3] BMT 2019 Discrete #9

Answers

1. Exercise 1: $\boxed{\frac{2025}{2023}}$

2. Exercise 2: $\boxed{14}$