Symmetric Rational Sums

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A sum of the same fraction over all the roots? Use $\frac{P'(x)}{P(x)} = \sum_i \frac{1}{x-r_i}$ to collapse the whole sum into one evaluation of P and P'.

You are given a polynomial P with roots $r_1, \ldots, r_n$ and asked for a *symmetric sum of rational expressions* in the roots — meaning you plug each root into the same fraction, like $\frac{1}{1-r}$, and add the results: $\sum_{i=1}^{n} \frac{1}{1-r_i}$. Clearing all those denominators by hand is hopeless when n is large, so the core move is the *logarithmic derivative*: $\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x-r_i}$, which collapses the entire sum into one evaluation of P and its derivative. When the fraction doesn't match this shape directly, first split it by partial fractions into pieces that do.

Problem 1 (BMT 2021/7)**.** Let $z_1, \ldots, z_{2020}$ be the roots of $z^{2020} + z^{2019} + \cdots + z + 1$. Compute $\sum_{i=1}^{2020} \frac{1}{1-z_i}$.

Solution. With $P(z) = z^{2020} + \cdots + 1$, the sum is exactly $P'(1)/P(1)$. Now $P(1) = 2021$ and $P'(1) = 2020 + 2019 + \cdots + 1 = \frac{2020 \cdot 2021}{2}$, so the sum is $\frac{2020 \cdot 2021/2}{2021} = 1010$. □

> **Remark.** When the denominator doesn't come from a single shift, split it first: $\frac{1}{1-r^2} = \frac{1}{2}\left(\frac{1}{1-r} + \frac{1}{1+r}\right)$, then apply Vieta to the shifted polynomials $P(1-x)$ and $P(x-1)$ — the first exercise below is exactly this.

Exercise 2 (BMT 2022/7)**.** Let r, s, t be the distinct roots of $x^3 - 2022x^2 + 2022x + 2022$. Compute $\frac{1}{1-r^2} + \frac{1}{1-s^2} + \frac{1}{1-t^2}$. Hint: 1

Exercise 3 (BMT 2019 Discrete #9)**.** Let $z = \frac{\sqrt{2}+i\sqrt{2}}{2}$. Given that $\sum_{k=0}^{13} \frac{1}{1 - ze^{ik\pi/7}} = a - bi$ with real numbers a and b, find $a + b$. Hint: 2

Answers: $\frac{2025}{2023}$; 14.

Hints

1. Split $\frac{1}{1-r^2} = \frac{1}{2}\left(\frac{1}{1-r} + \frac{1}{1+r}\right)$. The first sum is $P'(1)/P(1)$ as in the example; for the second, use the same move on $Q(x) = P(-x)$, whose roots are $-r, -s, -t$.

2. The 14 numbers $\omega_k = e^{ik\pi/7}$ are the roots of $P(t) = t^{14} - 1$. Write $\frac{1}{1-z\omega_k} = \frac{1/z}{1/z-\omega_k}$ and apply the logarithmic derivative to P at $x = 1/z$; then compute z^{14}.