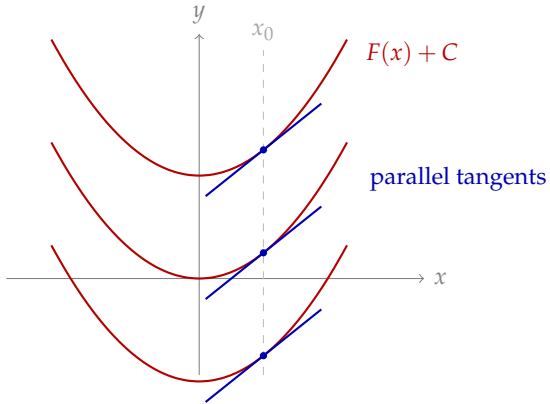
Indefinite Integration

David Liu



Same slope at every x — antiderivatives differ by a constant.

Overview

An antiderivative is a pattern match: find the substitution, the product, the partial fractions, or the trig form.

Overview: Read the Integrand First

Every integral below is a pattern match. Look at the integrand, find the cue, make the move.

Cue in the integrand	**Move**
an inner function and its derivative	u-substitution
a product: polynomial or $\ln x$ times e^x, $\sin x$	integration by parts
a ratio of polynomials	partial fractions
powers of $\sin x$ and $\cos x$	peel one off, or half-angle
$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$	trig substitution
numerator $=$ derivative of denominator	answer is $\ln \lvert f(x) \rvert$

Why this works: each move is a differentiation rule read backwards.

Formulas You Must Own

Differentiation owns four rules and five key derivatives. Integration is those rules read backwards — with one honest warning.

Formulas You Must Own: Differentiation Rules

- Power: $(x^n)' = n\,x^{n-1}$

Formulas You Must Own: Differentiation Rules

- Power: $(x^n)' = n\,x^{n-1}$
- Product: $(f\,g)' = f'\,g + f\,g'$

Formulas You Must Own: Differentiation Rules

- Power: $(x^n)' = n\,x^{n-1}$
- Product: $(f\,g)' = f'\,g + f\,g'$
- Quotient: $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$

Formulas You Must Own: Differentiation Rules

- Power: $(x^n)' = n\,x^{n-1}$
- Product: $(f\,g)' = f'\,g + f\,g'$
- Quotient: $\left(\dfrac{f}{g}\right)' = \dfrac{f'\,g - f\,g'}{g^2}$
- Chain: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Formulas You Must Own: Key Derivatives

- $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x$

Formulas You Must Own: Key Derivatives

- $(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x$
- $(e^x)' = e^x, \quad (\ln x)' = \dfrac{1}{x}$

Formulas You Must Own: Inverse Trig Derivatives

- $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $\quad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

Formulas You Must Own: Inverse Trig Derivatives

- $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $\quad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$
- $(\arctan x)' = \dfrac{1}{1+x^2}$

Formulas You Must Own: Inverse Trig Derivatives

- $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$
- $(\arctan x)' = \dfrac{1}{1+x^2}$
- Backwards: $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \quad \int \tfrac{dx}{1+x^2} = \arctan x + C$

Formulas You Must Own: Integration in Reverse

- Power rule, backwards: $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

Formulas You Must Own: Integration in Reverse

- Power rule, backwards: $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int e^x \, dx = e^x + C, \quad \int \frac{dx}{x} = \ln|x| + C$

Formulas You Must Own: Integration in Reverse

- Power rule, backwards: $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int e^x \, dx = e^x + C, \quad \int \frac{dx}{x} = \ln|x| + C$
- $\int \sin x \, dx = -\cos x + C, \quad \int \cos x \, dx = \sin x + C$

Formulas You Must Own: The Honest Warning

- There is **no** product rule and **no** quotient rule for integrals: $\int f \cdot g \, dx$ and $\int \frac{f}{g} \, dx$ have no formula.

Formulas You Must Own: The Honest Warning

- There is **no** product rule and **no** quotient rule for integrals: $\int f \cdot g \, dx$ and $\int \frac{f}{g} \, dx$ have no formula.
- There is **no** chain rule for integrals: $\int f(g(x)) \, dx$ has no formula.

Formulas You Must Own: The Honest Warning

- There is **no** product rule and **no** quotient rule for integrals: $\int f \cdot g \, dx$ and $\int \frac{f}{g} \, dx$ have no formula.
- There is **no** chain rule for integrals: $\int f(g(x)) \, dx$ has no formula.
- That is **why** the six techniques in this unit exist: u-substitution is the chain rule in reverse; integration by parts is the product rule in reverse.

Formulas You Must Own: Taylor Series

General: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (Maclaurin at $a = 0$)

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Formulas You Must Own: Taylor Series

General: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (Maclaurin at $a = 0$)

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Formulas You Must Own: Taylor Series

General: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (Maclaurin at $a = 0$)

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ $(-1 < x \leq 1)$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $(|x| < 1)$

Formulas You Must Own: Taylor Series

General: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (Maclaurin at $a = 0$)

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ $(-1 < x \leq 1)$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $(|x| < 1)$
- $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ $(|x| \leq 1)$

u-Substitution

Spot an inner function $g(x)$ whose derivative (up to a constant) multiplies the rest; set $u = g(x)$.

u-Substitution: The Pattern

Spot an inner function $g(x)$ whose derivative (up to a constant) multiplies the rest; set $u = g(x)$.

Chain rule in reverse: if the integrand looks like $f(g(x))\,g'(x)$, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

Roots of x clear by a power substitution ($x = u^n$); towers of exponentials are just iterated chain rules.

u-Substitution: Example

Compute

$$\int \frac{dx}{x+\sqrt{x}}.$$

1

[1] SMT 2025 Integration Bee #1

u-Substitution: Solution

Why $x = u^2$? It is the one substitution that turns $\sqrt{x}$ into a polynomial. With $x = u^2$, $dx = 2u\,du$:

u-Substitution: Solution

Why $x = u^2$? It is the one substitution that turns $\sqrt{x}$ into a polynomial. With $x = u^2$, $dx = 2u\,du$:

$$\int \frac{dx}{x + \sqrt{x}} = \int \frac{2u\,du}{u^2 + u} = \int \frac{2\,du}{u + 1}.$$

u-Substitution: Solution

Why $x = u^2$? It is the one substitution that turns $\sqrt{x}$ into a polynomial. With $x = u^2$, $dx = 2u\,du$:

$$\int \frac{dx}{x + \sqrt{x}} = \int \frac{2u\,du}{u^2 + u} = \int \frac{2\,du}{u + 1}.$$

Cancel the common factor u; what is left is $\frac{f'}{f}$ with $f = u + 1$:

$$= 2\ln|u + 1| + C = 2\ln|\sqrt{x} + 1| + C.$$

The substitution killed both x and $\sqrt{x}$ at once.

u-Substitution: Exercises 1–2

1. [HMMT 2009/2] $F : \mathbb{R} \to \mathbb{R}$ differentiable, $F(0) = -1$,

$$F'(x) = \sin(\sin(\sin(\sin x))) \cos(\sin(\sin x)) \cos(\sin x) \cos x.$$

   Find $F(x)$. *Hint:* $u = \sin(\sin(\sin x))$*; the rest is* $\sin u \, du$.

2. [HMMT 2006/5] $\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$. *Hint:* $x = u^6$ *clears both roots.*

u-Substitution: Exercises 3–4

3. [BMT Integration Bee 2024, qualifier] $\int_0^1 e^{e^{e^{e^x}}} \cdot e^{e^{e^x}} \cdot e^{e^x} \cdot e^x \, dx$. *Hint: the integrand is the derivative of a four-high power tower of e's.*

4. [SMT 2025 Integration Bee #2] $\int_0^\infty \frac{4x}{1+x^4} \, dx$. *Hint: $u = x^2$.*

Integration by Parts

$\int u\,dv = uv - \int v\,du$. *Differentiate the part that simplifies (polynomials,* $\ln x$*); integrate the part that cycles (*e^x*,* $\sin x$*,* $\cos x$*).*

Integration by Parts: The Pattern

$\int u\,dv = uv - \int v\,du$. Differentiate the part that simplifies (polynomials, $\ln x$); integrate the part that cycles (e^x, $\sin x$, $\cos x$).

Two classic shapes:

- **Boomerang:** parts twice and the original integral reappears — solve for it algebraically.
- $(f + f')e^x$**:** $\int (f + f')e^x\,dx = fe^x + C$ — spot the f and read the answer off.

Integration by Parts: Example

Compute

$$\int e^{3x} \sin x \, dx.$$

2

[2] SMT 2008/6

Integration by Parts: Solution

Take $u = \sin x$, $dv = e^{3x}\,dx$ — differentiate the trig, integrate the exponential. Let $I = \int e^{3x} \sin x\,dx$. Parts twice:

$$\begin{aligned} I &= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx \\ &= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \frac{1}{9}I. \end{aligned}$$

Integration by Parts: Solution

Take $u = \sin x$, $dv = e^{3x}\,dx$ — differentiate the trig, integrate the exponential. Let $I = \int e^{3x} \sin x\,dx$. Parts twice:

$$\begin{aligned} I &= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx \\ &= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \frac{1}{9}I. \end{aligned}$$

Hence $\frac{10}{9}I = \frac{e^{3x}}{9}(3\sin x - \cos x)$, so

Integration by Parts: Solution

Take $u = \sin x$, $dv = e^{3x}\,dx$ — differentiate the trig, integrate the exponential. Let $I = \int e^{3x} \sin x\,dx$. Parts twice:

$$\begin{aligned} I &= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx \\ &= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \frac{1}{9}I. \end{aligned}$$

Hence $\frac{10}{9}I = \frac{e^{3x}}{9}(3\sin x - \cos x)$, so

$$\int e^{3x}\sin x\,dx = \frac{e^{3x}}{10}(3\sin x - \cos x) + C.$$

The boomerang: I came back, and we solved for it.

Integration by Parts: Exercises 1–3

1. [SMT 2009/4] $\int_0^1 \tan^{-1} x\,dx$. *Hint:* $u = \tan^{-1} x$, $dv = dx$; *the leftover is* f'/f.

2. [HMMT 2008/3] Find all $y > 1$ with $\int_1^y x \ln x\,dx = \frac{1}{4}$. *Hint: parts with* $u = \ln x$; *factor the resulting equation in* y.

3. [BMT Integration Bee] $\int \frac{x-1}{(x+1)^3} e^x\,dx$. *Hint: for any differentiable* f, $\int (f + f')e^x\,dx = fe^x + C$. *Find* f.

Integration by Parts: Exercises 4–5 (continued)

4. [SMT 2013/4] $\int_0^4 e^{\sqrt{x}}\,dx$. *Hint: $w = \sqrt{x}$; then integrate by parts.*

5. [MIT Integration Bee 2011, Qualifying #11] $\int \cos(\log x)\,dx$. *Hint: integrate by parts twice; the original integral reappears.*

Partial Fractions

For a proper rational function, factor the denominator and split into simple pieces. If the numerator's degree is at least the denominator's, divide first.

Partial Fractions: The Pattern

For a proper rational function, factor the denominator and split into simple pieces. If the numerator's degree is at least the denominator's, divide first.

Over $\mathbb{R}$, every piece is one of

$$\frac{A}{x-r}, \qquad \frac{Bx+C}{x^2+px+q},$$

and a sum of logs is secretly the log of a product — so a messy antiderivative may collapse when the problem asks for e^A instead of A.

Partial Fractions: Example

Compute e^A, where

$$A = \int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1}\, dx.$$

3

[3] HMMT 2009/3

Partial Fractions: Try it

Compute e^A, where

$$A = \int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1}\, dx.$$

Hint: How would you factor $x^3 + x^2 + x + 1$*?* [4]

[4] HMMT 2009/3

Partial Fractions: Solution

$x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ (group: $x^2(x+1) + (x+1)$) and

$$\frac{2x^2 + x + 1}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{x}{x^2+1},$$

where matching coefficients in $\frac{P}{x+1} + \frac{Qx+R}{x^2+1}$ gives $P = 1$, $Q = 1$, $R = 0$ (or check by re-adding), so

$$A = \left[\ln\left((x+1)\sqrt{x^2+1}\right)\right]_{3/4}^{4/3} = \ln\frac{35/9}{35/16} = \ln\frac{16}{9}.$$

Thus $e^A = \frac{16}{9}$. The logs recombined into a single ratio.

Partial Fractions: Exercises

1. [MIT Integration Bee 2025, Regular Season #4] $\int \frac{dx}{1-x^4}$. *Hint:* $1 - x^4 = (1 - x)(1 + x)(1 + x^2)$.

2. [SMT 2018/7] $\int_{-2}^{0} \frac{x^3+4x^2+7x-20}{x^2+4x+8}\,dx$ $+ \int_{0}^{2} \frac{2x^3-7x^2+9x-10}{x^2+4}\,dx$. *Hint: shift the first integral by* $u = x + 2$ *so both share denominator* $u^2 + 4$*; add, then divide.*

Powers of Sine and Cosine

Odd power: peel off one factor and substitute $u = \cos x$ *(or* $\sin x$*). Even power: use the half-angle formulas. Products often collapse via* $\sin 2x = 2 \sin x \cos x$.

Powers of Sine and Cosine: The Pattern

Odd power: peel off one factor and substitute $u = \cos x$ (or $\sin x$). Even power: use the half-angle formulas.

Products of sines and cosines often collapse via $\sin 2x = 2 \sin x \cos x$, and product-to-sum identities turn $\cos A \cos B$ into a sum of single cosines.

Powers of Sine and Cosine: Example

Compute

$$\int_0^{\pi/4} (\cos x - 2 \sin x \sin 2x)\, dx.$$

5

[5] SMT 2019/3

Powers of Sine and Cosine: Solution

Since $\sin 2x = 2 \sin x \cos x$, we have $2 \sin x \cdot \sin 2x = 4 \sin^2 x \cos x$.

Thus the integrand is $\cos x(1 - 4 \sin^2 x)$. With $u = \sin x$:

Powers of Sine and Cosine: Solution

Since $\sin 2x = 2\sin x \cos x$, we have $2\sin x \cdot \sin 2x = 4\sin^2 x \cos x$.

Thus the integrand is $\cos x(1 - 4\sin^2 x)$. With $u = \sin x$:

$$\int_0^{\pi/4} \cos x(1 - 4\sin^2 x)\, dx = \int_0^{\sqrt{2}/2} (1 - 4u^2)\, du$$

Powers of Sine and Cosine: Solution

Since $\sin 2x = 2\sin x \cos x$, we have $2\sin x \cdot \sin 2x = 4\sin^2 x \cos x$.

Thus the integrand is $\cos x(1 - 4\sin^2 x)$. With $u = \sin x$:

$$\int_0^{\pi/4} \cos x(1 - 4\sin^2 x)\, dx = \int_0^{\sqrt{2}/2} (1 - 4u^2)\, du$$

$$= \left[u - \frac{4u^3}{3}\right]_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{6}.$$

Powers of Sine and Cosine: Exercises

1. [SMT 2025 Integration Bee #5] $\int \frac{\sin^4\theta}{\cos\theta}\,d\theta$. *Hint:* $\sin^4\theta = (1-\cos^2\theta)^2$; *expand and split into* $\sec\theta$, $\cos\theta$, $\cos^3\theta$.

2. [MIT Integration Bee 2025, Regular Season #5] $\int_{-\pi/2}^{\pi/2} \cos(20x)\cos(25x)\,dx$. *Hint:* $\cos A\cos B = \frac{1}{2}\big[\cos(A-B)+\cos(A+B)\big]$.

Trigonometric Substitution

$\sqrt{a^2 - x^2}$ *calls for* $x = a \sin\theta$; $\sqrt{a^2 + x^2}$ *for* $x = a \tan\theta$; $\sqrt{x^2 - a^2}$ *for* $x = a \sec\theta$.

Trigonometric Substitution: The Pattern

$\sqrt{a^2 - x^2}$ calls for $x = a \sin\theta$; $\sqrt{a^2 + x^2}$ for $x = a \tan\theta$; $\sqrt{x^2 - a^2}$ for $x = a \sec\theta$.

The root is an invitation to make it a trig function. Track the bounds through the substitution — an improper bound may become a clean $\theta = \pi/2$.

Trigonometric Substitution: Example

Compute

$$\int_1^\infty \frac{dx}{x^2\sqrt{x^2-1}}.$$

6

[6] SMT 2025 Integration Bee #3

Trigonometric Substitution: Solution

Set $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$; then $\sqrt{x^2-1} = \tan\theta$, taking $\theta \in [0, \pi/2)$, so $\tan\theta \geq 0$ and there is no sign ambiguity.

Trigonometric Substitution: Solution

Set $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$; then $\sqrt{x^2-1} = \tan\theta$, taking $\theta \in [0, \pi/2)$, so $\tan\theta \geq 0$ and there is no sign ambiguity.

$$\frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \cos\theta\,d\theta.$$

Trigonometric Substitution: Solution

Set $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$; then $\sqrt{x^2-1} = \tan\theta$, taking $\theta \in [0, \pi/2)$, so $\tan\theta \geq 0$ and there is no sign ambiguity.

$$\frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \cos\theta\,d\theta.$$

As x runs $1 \to \infty$, θ runs $0 \to \pi/2$, so the integral is

$$\int_0^{\pi/2} \cos\theta\,d\theta = 1.$$

Trigonometric Substitution: Exercises

1. [MIT Integration Bee 2011, Qualifying #21] $\int_0^6 \sqrt{6x - x^2}\, dx$. *Hint: complete the square; the graph is a semicircle.*

2. [MIT Integration Bee 2011, Qualifying #10] $\int \frac{dx}{(x^2+1)^2}$. *Hint:* $x = \tan\theta$.

The Logarithmic Derivative

If the numerator is (a constant multiple of) the derivative of the denominator, the answer is a logarithm: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

The Logarithmic Derivative: The Pattern

If the numerator is (a constant multiple of) the derivative of the denominator, the answer is a logarithm: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

Disguised versions abound: divide by e^x to expose the derivative, sum a geometric series first, or write the numerator as a combination of $\frac{g(x)}{x}$ and $g'(x)$.

The Logarithmic Derivative: Example

Let

$$T = \int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1}\, dx.$$

Evaluate e^T. [7]

[7] HMMT 2008/8

The Logarithmic Derivative: Try it

Let

$$T = \int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1}\, dx.$$

Evaluate e^T.

What operation would turn the numerator into the derivative of the denominator? [8]

[8] HMMT 2008/8

The Logarithmic Derivative: Solution

Divide top and bottom by e^x: the integrand becomes $\frac{2e^{2x}+e^x-e^{-x}}{e^{2x}+e^x-1+e^{-x}}$, and the numerator is exactly the derivative of the denominator — check: $\frac{d}{dx}[e^{2x}+e^x-1+e^{-x}] = 2e^{2x}+e^x-e^{-x}$ ✓.

The Logarithmic Derivative: Solution

Divide top and bottom by e^x: the integrand becomes $\frac{2e^{2x}+e^x-e^{-x}}{e^{2x}+e^x-1+e^{-x}}$, and the numerator is exactly the derivative of the denominator — check: $\frac{d}{dx}\left[e^{2x}+e^x-1+e^{-x}\right] = 2e^{2x}+e^x-e^{-x}$ ✓. Thus

$$T = \Big[\ln\left(e^{2x}+e^x-1+e^{-x}\right)\Big]_0^{\ln 2} = \ln\frac{11/2}{2} = \ln\frac{11}{4},$$

so $e^T = \frac{11}{4}$.

The Logarithmic Derivative: Exercises 1–2

1. [SMT 2025 Integration Bee #9] $\int \frac{x+1}{x^2+x\ln x}\,dx$. *Hint:* $u = x + \ln x$.

2. [HMMT 1998/4] Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots$ for $-1 \le x \le 1$. Find $e^{\int_0^1 f(x)\,dx}$. *Hint: sum the geometric series first; then the integrand is f'/f-shaped.*

The Logarithmic Derivative: Exercises 3–4 (continued)

3. [MIT Integration Bee 2024, Qualifying #4] $\int \frac{dx}{x \log x + 2x}$. *Hint: factor the denominator; then* $u = \log x$.

4. [HMMT 2007, Guts #33] $\int_1^2 \frac{9x+4}{x^5+3x^2+x}\,dx$. *Hint: let* g *be the denominator; write the numerator as a combination of* $g(x)/x$ *and* $g'(x)$.

Challenge

Differentiate a guess and match exponents — the answer can be read off without integrating.

Differentiate and Match Exponents: Challenge

For some constants a, b, c,

$$\int \frac{\sin^a x - \cos^a x}{\sin^b x \cos^b x}\,dx = \frac{\csc^c x}{c} + \frac{\sec^c x}{c} + K.$$

Given $a = 2021$, find $a + b + c$.

Hint: differentiate the right side and match exponents of $\sin x$ *and* $\cos x$ *with the left.* [9]

[9] BMT 2021/5

Answers

All answers, omitting $+C$.

Answers (1/3)

Items 1–6 are the worked examples, then the exercises, in order.

1. $2\ln|\sqrt{x}+1|$
2. $\frac{e^{3x}}{10}(3\sin x - \cos x)$
3. $\frac{16}{9}$
4. $\frac{\sqrt{2}}{6}$
5. 1
6. $\frac{11}{4}$
7. $-\cos(\sin(\sin(\sin x)))$
8. $5 - 6\ln 2$

Answers (2/3)

9. $e^{e^{e^e}} - e^{e^{e^e}}$
10. π
11. $\frac{\pi}{4} - \frac{1}{2}\ln 2$
12. $\sqrt{e}$
13. $\frac{e^x}{(x+1)^2}$
14. $2e^2 + 2$
15. $\frac{x}{2}\big(\cos(\log x) + \sin(\log x)\big)$

Answers (3/3)

16. $\frac{1}{2}\arctan x + \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|$

17. -12

18. $-\frac{\sin^3\theta}{3} - \sin\theta + \ln|\sec\theta + \tan\theta|$

19. $\frac{2}{9}$

20. $\frac{9\pi}{2}$

21. $\frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}$

22. $\ln|x + \ln x|$

23. 4

24. $\ln|\log x + 2|$

25. $\ln\frac{80}{23}$

26. 6060