Indefinite Integration

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An antiderivative is a pattern match: find the substitution, the product, the partial fractions, or the trig form. Every problem below is a real contest problem.

1. u-substitution

Spot an inner function $g(x)$ whose derivative (up to a constant) multiplies the rest; set $u = g(x)$.

Problem 1 (SMT 2025 Integration Bee #1)**.** Compute $\int \frac{dx}{x+\sqrt{x}}$.

Solution. With $x = u^2$, $dx = 2u\,du$:

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = \int \frac{2\,du}{u+1} = 2\ln|u+1| + C = 2\ln|\sqrt{x}+1| + C.$$

□

Exercise 2 (HMMT 2009/2)**.** $F : \mathbb{R} \to \mathbb{R}$ is differentiable, $F(0) = -1$, and

$$F'(x) = \sin(\sin(\sin(\sin x)))\cos(\sin(\sin x))\cos(\sin x)\cos x.$$

Find $F(x)$. Hint: 1

Exercise 3 (HMMT 2006/5)**.** Compute $\int_0^1 \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$. Hint: 2

Exercise 4 (BMT Integration Bee 2024, qualifier)**.** Compute $\int_0^1 e^{e^{e^{e^x}}} \cdot e^{e^{e^x}} \cdot e^{e^x} \cdot e^x\,dx$. Hint: 3

Exercise 5 (SMT 2025 Integration Bee #2)**.** Compute $\int_0^\infty \frac{4x}{1+x^4}\,dx$. Hint: 4

2. Integration by parts

$\int u\,dv = uv - \int v\,du$. Differentiate the part that simplifies (polynomials, $\ln x$); integrate the part that cycles (e^x, $\sin x$, $\cos x$).

Problem 6 (SMT 2008/6)**.** Compute $\int e^{3x}\sin x\,dx$.

Solution. Let $I = \int e^{3x}\sin x\,dx$. Parts twice:

$$I = \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx = \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \frac{1}{9}I.$$

Hence $\frac{10}{9} I = \frac{e^{3x}}{9}(3\sin x - \cos x)$, so

$$\int e^{3x} \sin x \, dx = \frac{e^{3x}}{10}(3\sin x - \cos x) + C.$$

□

Exercise 7 (SMT 2009/4). Compute $\int_0^1 \tan^{-1} x \, dx$. Hint: 5

Exercise 8 (HMMT 2008/3). Find all $y > 1$ with $\int_1^y x \ln x \, dx = \frac{1}{4}$. Hint: 6

Exercise 9 (BMT Integration Bee). Compute $\int \frac{x-1}{(x+1)^3} e^x \, dx$. Hint: 7

Exercise 10 (SMT 2013/4). Compute $\int_0^4 e^{\sqrt{x}} \, dx$. Hint: 8

Exercise 11 (MIT Integration Bee 2011, Qualifying #11). Compute $\int \cos(\log x) \, dx$. Hint: 9

3. Partial fractions

For a proper rational function, factor the denominator and split into simple pieces. If the numerator's degree is at least the denominator's, divide first.

Problem 12 (HMMT 2009/3). Compute e^A, where $A = \int_{3/4}^{4/3} \frac{2x^2+x+1}{x^3+x^2+x+1} \, dx$.

Solution. $x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ and

$$\frac{2x^2+x+1}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{x}{x^2+1},$$

so $A = \left[\ln\left((x+1)\sqrt{x^2+1}\right)\right]_{3/4}^{4/3} = \ln \frac{35/9}{35/16} = \ln \frac{16}{9}$. Thus $e^A = \frac{16}{9}$. □

Exercise 13 (MIT Integration Bee 2025, Regular Season #4). Compute $\int \frac{dx}{1-x^4}$. Hint: 10

Exercise 14 (SMT 2018/7). Compute $\int_{-2}^{0} \frac{x^3+4x^2+7x-20}{x^2+4x+8} \, dx + \int_0^2 \frac{2x^3-7x^2+9x-10}{x^2+4} \, dx$. Hint: 11

4. Powers of sine and cosine

Odd power: peel off one factor and substitute $u = \cos x$ (or $\sin x$). Even power: use the half-angle formulas. Products of sines and cosines often collapse via $\sin 2x = 2\sin x \cos x$.

Problem 15 (SMT 2019/3). Compute $\int_0^{\pi/4} (\cos x - 2\sin x \sin 2x) \, dx$.

Solution. $2\sin x \sin 2x = 4\sin^2 x \cos x$, so the integrand is $\cos x(1 - 4\sin^2 x)$. With $u = \sin x$:

$$\int_0^{\pi/4} \cos x(1 - 4\sin^2 x) \, dx = \int_0^{\sqrt{2}/2} (1 - 4u^2) \, du = \left[u - \frac{4u^3}{3}\right]_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{6}.$$

□

Exercise 16 (SMT 2025 Integration Bee #5)**.** Compute $\int \frac{\sin^4\theta}{\cos\theta}\,d\theta$. Hint: 12

Exercise 17 (MIT Integration Bee 2025, Regular Season #5)**.** Compute $\int_{-\pi/2}^{\pi/2} \cos(20x)\cos(25x)\,dx$. Hint: 13

5. Trigonometric substitution

$\sqrt{a^2-x^2}$ calls for $x = a\sin\theta$; $\sqrt{a^2+x^2}$ for $x = a\tan\theta$; $\sqrt{x^2-a^2}$ for $x = a\sec\theta$.

Problem 18 (SMT 2025 Integration Bee #3)**.** Compute $\int_1^\infty \frac{dx}{x^2\sqrt{x^2-1}}$.

Solution. Set $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$; then $\sqrt{x^2-1} = \tan\theta$ and

$$\frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \cos\theta\,d\theta.$$

As x runs $1 \to \infty$, θ runs $0 \to \pi/2$, so the integral is $\int_0^{\pi/2}\cos\theta\,d\theta = 1$. □

Exercise 19 (MIT Integration Bee 2011, Qualifying #21)**.** Compute $\int_0^6 \sqrt{6x-x^2}\,dx$. Hint: 14

Exercise 20 (MIT Integration Bee 2011, Qualifying #10)**.** Compute $\int \frac{dx}{(x^2+1)^2}$. Hint: 15

6. The logarithmic derivative

If the numerator is (a constant multiple of) the derivative of the denominator, the answer is a logarithm: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

Problem 21 (HMMT 2008/8)**.** Let $T = \int_0^{\ln 2} \frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1}\,dx$. Evaluate e^T.

Solution. Divide top and bottom by e^x: the integrand becomes $\frac{2e^{2x}+e^x-e^{-x}}{e^{2x}+e^x-1+e^{-x}}$, and the numerator is exactly the derivative of the denominator. Thus

$$T = \left[\ln\left(e^{2x}+e^x-1+e^{-x}\right)\right]_0^{\ln 2} = \ln\frac{11/2}{2} = \ln\frac{11}{4},$$

so $e^T = \frac{11}{4}$. □

Exercise 22 (SMT 2025 Integration Bee #9)**.** Compute $\int \frac{x+1}{x^2+x\ln x}\,dx$. Hint: 16

Exercise 23 (HMMT 1998/4)**.** Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots$ for $-1 \le x \le 1$. Find $e^{\int_0^1 f(x)\,dx}$. Hint: 17

Exercise 24 (MIT Integration Bee 2024, Qualifying #4)**.** Compute $\int \frac{dx}{x\log x+2x}$. Hint: 18

Exercise 25 (HMMT 2007, Guts #33)**.** Compute $\int_1^2 \frac{9x+4}{x^5+3x^2+x}\,dx$. Hint: 19

Challenge

Exercise 26 (BMT 2021/5)**.** For some constants a, b, c,

$$\int \frac{\sin^a x - \cos^a x}{\sin^b x \cos^b x}\,dx = \frac{\csc^c x}{c} + \frac{\sec^c x}{c} + K.$$

Given $a = 2021$, find $a + b + c$. Hint: 20

Answers (omitting $+C$): $2\ln|\sqrt{x}+1|$; $\frac{e^{3x}}{10}(3\sin x - \cos x)$; $\frac{16}{9}$; $\frac{\sqrt{2}}{6}$; 1; $\frac{11}{4}$; $-\cos(\sin(\sin(\sin x)))$; $5 - 6\ln 2$; $e^{e^{e^e}} - e^{e^e}$; π; $\frac{\pi}{4} - \frac{1}{2}\ln 2$; $\sqrt{e}$; $\frac{e^x}{(x+1)^2}$; $2e^2 + 2$; $\frac{x}{2}\big(\cos(\log x) + \sin(\log x)\big)$.

Answers, continued: $\frac{1}{2}\arctan x + \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|$; -12; $-\frac{\sin^3\theta}{3} - \sin\theta + \ln|\sec\theta + \tan\theta|$; $\frac{2}{9}$; $\frac{9\pi}{2}$; $\frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}$; $\ln|x + \ln x|$; 4; $\ln|\log x + 2|$; $\ln\frac{80}{23}$; 6060.

Hints

1. $u = \sin(\sin(\sin x))$; the rest is $\sin u\,du$.
2. $x = u^6$ clears both roots.
3. The integrand is the derivative of a four-high power tower of e's.
4. $u = x^2$.
5. $u = \tan^{-1} x$, $dv = dx$; the leftover is f'/f.
6. Parts with $u = \ln x$; factor the resulting equation in y.
7. For any differentiable f, $\int (f + f')e^x\,dx = fe^x + C$. Find f.
8. $w = \sqrt{x}$; then integrate by parts.
9. Integrate by parts twice; the original integral reappears.
10. $1 - x^4 = (1-x)(1+x)(1+x^2)$.
11. Shift the first integral by $u = x + 2$ so both share denominator $u^2 + 4$; add, then divide.
12. $\sin^4\theta = (1 - \cos^2\theta)^2$; expand and split into $\sec\theta$, $\cos\theta$, $\cos^3\theta$.
13. $\cos A\cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$.
14. Complete the square; the graph is a semicircle.
15. $x = \tan\theta$.
16. $u = x + \ln x$.
17. Sum the geometric series first; then the integrand is f'/f-shaped.

18. Factor the denominator; then $u = \log x$.

19. Let g be the denominator; write the numerator as a combination of $g(x)/x$ and $g'(x)$.

20. Differentiate the right side and match exponents of $\sin x$ and $\cos x$ with the left.