Higher Derivatives via Patterns

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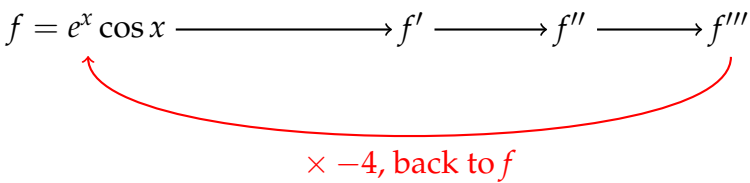
The key idea: differentiate until the derivatives repeat — then jump straight to the one you want.



$$f^{(4)} = -4f, \text{ so } f^{(100)} = (-4)^{25} f.$$

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Derivative Cycling

Don't differentiate a hundred times — find how the derivatives repeat, then jump straight to the one you want.

Derivative cycling: the pattern

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Differentiating $e^x \cos x$ only ever produces combinations of $e^x \cos x$ and $e^x \sin x$, so it is worth looking for a repeat: here $f^{(4)} = -4f$ (four direct differentiations, checked on the next slides).

Compute $f', f'', f''', \ldots$ until they repeat up to a constant factor, say $f^{(k)} = cf$. Then every k-th derivative is a power of c times f, and for $n = qk + r$ — apply the k-cycle q times —

$$\boxed{f^{(n)} = c^q f^{(r)}}$$

where $n = qk + r$, i.e. $q = \lfloor n/k \rfloor$, $r = n \bmod k$. No repeated differentiating needed — just reduce n modulo the period k.

Derivative cycling: Example

What is the 100th derivative of $f(x) = e^x \cos x$ at $x = \pi$?

Strategy: differentiate a few times, find the repeat, jump straight to $n = 100$. [1]

[1] BMT 2018/3

Derivative cycling: Solution

Compute the first few derivatives:

$$f'(x) = e^x(\cos x - \sin x),$$

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The cycle has period 4 with factor $c = -4$. *Why stop here? $f^{(4)}$ is a multiple of f, so the cycle has closed.*

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$$f^{(100)}(x) = (-4)^{25} f(x).$$

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$$f^{(100)}(\pi) = (-4)^{25} \cdot (-e^\pi) = \boxed{4^{25} e^\pi}.$$

Series Recognition

Get the derivative formula once, then read the infinite sum as a Taylor series you already know.

Series recognition: the setup

Two series facts, used constantly:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \quad (|x| < 1).$$

The move: find a closed formula for the derivatives (often by induction), substitute it, and recognize the resulting sum as one of these Taylor series.

Series recognition: Example

For a positive number x, let $f_0(x) = \frac{1}{x}$ and $f_n(x) = \frac{d^n}{dx^n}\frac{1}{x}$ for all positive integers n. If

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{f_n(x)},$$

compute $g(1)$.

Strategy: formula for $f_n(x)$ first, then read the sum as a Taylor series. [2]

[2] BMT 2023/6

Series recognition: Try it

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Hint: Compute f_0, f_1, f_2 explicitly — what pattern do you see? [3]

[3] BMT 2023/6

Series recognition: Solution

Claim $f_n(x) = \dfrac{(-1)^n n!}{x^{n+1}}$. It holds for $n = 0$, and differentiating gives

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$$\frac{d}{dx}\frac{(-1)^n n!}{x^{n+1}} = \frac{(-1)^n n! \cdot (-(n+1))}{x^{n+2}} = \frac{(-1)^{n+1}(n+1)!}{x^{n+2}}.$$

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Then

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(-1)^n n!} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = xe^{-x},$$

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the Taylor series of e^{-x}. Hence $\boxed{g(1) = 1/e}$.

Taylor Coefficients

The Taylor series already knows every derivative at the center: read the coefficient, multiply by $n!$.

Taylor coefficients: the setup

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The Taylor series of f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The move: to find $f^{(n)}(a)$, expand the Taylor series, read off the coefficient of $(x-a)^n$, and multiply by $n!$. No differentiating needed.

Example: 100th derivative of e^{x^2} (statement)

What is the 100th derivative of $f(x) = e^{x^2}$ at $x = 0$?

Strategy: expand e^{x^2} as a Taylor series and read off the coefficient of x^{100}.

Example: 100th derivative of e^{x^2} (try it)

What is the 100th derivative of $f(x) = e^{x^2}$ at $x = 0$?

Hint: $e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$ — which term gives x^{100}?

Example: 100th derivative of e^{x^2} (solution)

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}.$$

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Why expand? $f^{(100)}(0)$ is 100! times the coefficient of x^{100}, so we only need one coefficient.

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Why expand? $f^{(100)}(0)$ is 100! times the coefficient of x^{100}, so we only need one coefficient.

The x^{100} term comes from $2k = 100$, i.e. $k = 50$, with coefficient $\frac{1}{50!}$. Hence

$$\boxed{f^{(100)}(0) = \frac{100!}{50!}}.$$

Practice Exercises

Three problems to try — the same moves you just saw.

Practice: derivative cycling

Exercise 1. Find the 50th derivative of $e^x \sin x$ at $x = 0$.

Hint: Compute $f', f'', f''', f^{(4)}$ and look for the repeat. Reduce 50 modulo the period.

Practice: series recognition

Exercise 2. Let a_n be the n-th derivative of $\ln(1+x)$ at $x=0$, for $n \geq 1$. Compute

$$\sum_{n=1}^{\infty} \frac{a_n}{n! \cdot 2^n}.$$

Hint: Find the formula for a_n first, then recognize the Taylor series of $\ln(1+x)$.

Practice: Taylor coefficients

Exercise 3. Find the 50th derivative of $x^2 \cos x$ at $x = 0$.

Hint: Expand $\cos x$ as its Taylor series, multiply by x^2, and read off the coefficient of x^{50}.

Answers

1. Exercise 1: $\boxed{2 \cdot 4^{12}}$
2. Exercise 2: $\boxed{\ln \frac{3}{2}}$
3. Exercise 3: $\boxed{2450}$