Higher Derivatives via Patterns

David Liu

Don't differentiate a hundred times — find how the derivatives repeat, then jump straight to the one you want.

Derivative cycling. Compute $f', f'', f''', \ldots$ until they repeat up to a constant factor, say $f^{(k)} = c\,f$. Then every k-th derivative is a power of c times f, and $f^{(n)} = c^{\lfloor n/k \rfloor} f^{(n \bmod k)}$ — no repeated differentiating needed.

We will use two series facts:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad (|x| < 1).$$

Taylor coefficients. The Taylor series of f at a is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ — so $f^{(n)}(a)$ is $n!$ times the coefficient of $(x-a)^n$. To find a high-order derivative at a point, expand the Taylor series and read off the coefficient; no differentiating needed.

Problem 1 (Example)**.** What is the 100th derivative of $f(x) = e^{x^2}$ at $x = 0$?

Solution. Expand as a Taylor series:

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}.$$

The x^{100} term comes from $2k = 100$, i.e. $k = 50$, with coefficient $\frac{1}{50!}$. Hence

$$f^{(100)}(0) = 100! \cdot \frac{1}{50!} = \frac{100!}{50!}.$$

□

Problem 2 (BMT 2018/3)**.** What is the 100th derivative of $f(x) = e^x \cos x$ at $x = \pi$?

Solution. Compute the first few derivatives:

$$\begin{aligned} f'(x) &= e^x(\cos x - \sin x), \\ f''(x) &= -2e^x \sin x, \\ f'''(x) &= -2e^x(\sin x + \cos x), \\ f^{(4)}(x) &= -4e^x \cos x = -4f(x). \end{aligned}$$

The derivatives cycle with period 4 up to the factor -4. Hence

$$f^{(100)}(x) = (-4)^{25} f(x) = -4^{25} f(x).$$

At $x = \pi$, $f(\pi) = e^{\pi} \cos \pi = -e^{\pi}$, so

$$f^{(100)}(\pi) = -4^{25} \cdot (-e^{\pi}) = 4^{25} e^{\pi}.$$

□

Problem 3 (BMT 2023/6)**.** For a positive number x, let $f_0(x) = \frac{1}{x}$ and $f_n(x) = \frac{d^n}{dx^n} \frac{1}{x}$ for all positive integers n. If

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{f_n(x)},$$

compute $g(1)$.

Solution. Find the derivative formula first. We claim $f_n(x) = \frac{(-1)^n n!}{x^{n+1}}$: it holds for $n = 0$, and differentiating gives

$$\frac{d}{dx} \frac{(-1)^n n!}{x^{n+1}} = \frac{(-1)^n n! \cdot (-(n+1))}{x^{n+2}} = \frac{(-1)^{n+1} (n+1)!}{x^{n+2}}.$$

Then

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(-1)^n n!} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = x e^{-x},$$

the Taylor series of e^{-x}. Hence $g(1) = \frac{1}{e}$. □

Exercise 4. Find the 50th derivative of $e^x \sin x$ at $x = 0$. Hint: 1

Exercise 5. Let a_n be the n-th derivative of $\ln(1 + x)$ at $x = 0$, for $n \geq 1$. Compute $\sum_{n=1}^{\infty} \frac{a_n}{n! \cdot 2^n}$. Hint: 2

Exercise 6. Find the 50th derivative of $x^2 \cos x$ at $x = 0$. Hint: 3

Answers: $\frac{100!}{50!}$; $4^{25} e^{\pi}$; $1/e$; $2 \cdot 4^{12}$; $\ln \frac{3}{2}$; 2450.

Hints

1. Compute $f', f'', f''', f^{(4)}$ and look for the repeat. Reduce 50 modulo the period.
2. Find the formula for a_n first, then recognize the Taylor series of $\ln(1 + x)$.
3. Expand $\cos x$ as its Taylor series, multiply by x^2, and read off the coefficient of x^{50}.