# Easy Problems

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Coming up:

$$\int_0^1 e^x \, e^{e^x} \, e^{e^{e^x}} \, dx$$

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# How to Use These Warm-Ups

Fourteen one-step warm-ups. Each takes a single idea.

## The move

Before computing anything, ask: *which one rule does this need?* A derivative rule, the FTC, a symmetry, a factorization, or the chain rule run backwards. Name it, apply it once, and stop.

## Why

Contest warm-ups reward spotting the idea; brute force on these usually means a missed shortcut.

# Second derivative of a polynomial

*Differentiate a polynomial term by term — the constant vanishes.*

# Second derivative of a polynomial

Let $f(x) = \dfrac{x^{2020}}{2020} + 2020!$. Compute $f''(1)$. [1]

[1] BMT 2020/1

# Second derivative of a polynomial

Let $f(x) = \dfrac{x^{2020}}{2020} + 2020!$. Compute $f''(1)$. [1]

$$f'(x) = x^{2019}, \qquad f''(x) = 2019x^{2018}$$

$$\boxed{f''(1) = 2019}$$

[1] BMT 2020/1

# Even integral

*Spot the even function and halve the work.*

# Even integral

Compute $\int_{-20}^{20} (20 - |x|)\, dx$. [2]

[2] BMT 2020/2

# Even integral

Compute $\int_{-20}^{20} (20 - |x|)\, dx$. [2] The integrand is even:

$$2\int_{0}^{20} (20 - x)\, dx = 2\left[20x - \frac{x^2}{2}\right]_{0}^{20} = 2(400 - 200) = \boxed{400}$$

[2] BMT 2020/2

# Fundamental theorem of calculus

*FTC part 1 differentiates the accumulated area back to the integrand.*

# Fundamental theorem of calculus

Let $g(x) = \int_{2021}^{x} (e^t - 2t)\, dt$. Compute $g'(2021)$. [3]

[3] BMT 2021/1

# Fundamental theorem of calculus

Let $g(x) = \int_{2021}^{x} (e^t - 2t)\, dt$. Compute $g'(2021)$. [3] By FTC,

$$g'(x) = e^x - 2x, \qquad \boxed{g'(2021) = e^{2021} - 4042}$$

[3] BMT 2021/1

# Higher derivative of a quartic

*A quartic's fourth derivative is just* 4! *times the leading coefficient.*

# Higher derivative of a quartic

Let $f(x) = (x+3)(2x+5)(3x+7)(x+1)$. Compute $f''''(5)$. [4]

[4] BMT 2021/2

# Higher derivative of a quartic

Let $f(x) = (x+3)(2x+5)(3x+7)(x+1)$. Compute $f''''(5)$. [4] $f$ is quartic with leading coefficient $1 \cdot 2 \cdot 3 \cdot 1 = 6$, so

$$f''''(x) = 6 \cdot 4! = \boxed{144} \quad \text{(constant)}$$

[4] BMT 2021/2

# Trig identity before differentiating

$\sin^2 x + \cos^2 x = 1$ *kills both terms before differentiating.*

# Trig identity before differentiating

Compute $\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 x$. [5]

[5] BMT 2022/1

# Trig identity before differentiating

Compute $\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 x$. [5] Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{d}{dx}(1) = \boxed{0}$$

[5] BMT 2022/1

# Trig identity before differentiating

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(Alternatively, differentiate term by term: $2\sin x \cos x - 2\sin x \cos x = 0$.)

[5] BMT 2022/1

# Reverse chain rule

*Recognize the chain rule running in reverse.*

# Reverse chain rule

Compute $\int_0^1 e^x \cdot e^{e^x} \cdot e^{e^{e^x}}\, dx$. [6]

[6] BMT 2022/3

# Reverse chain rule

Compute $\int_0^1 e^x \cdot e^{e^x} \cdot e^{e^{e^x}}\, dx$. [6]

Build the derivative from the inside out — each new factor is what the chain rule produces for one more layer.

$$\left(e^{e^x}\right)' = e^x \cdot e^{e^x}, \qquad \left(e^{e^{e^x}}\right)' = e^{e^{e^x}} \cdot \left(e^{e^x}\right)' = e^x \cdot e^{e^x} \cdot e^{e^{e^x}}.$$

So the integrand is exactly $\frac{d}{dx} e^{e^{e^x}}$:

[6] BMT 2022/3

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So the integrand is exactly $\frac{d}{dx} e^{e^{e^x}}$:

$$\left[e^{e^{e^x}}\right]_0^1 = e^{e^e} - e^e$$

$$\boxed{e^{e^e} - e^e}$$

[6] BMT 2022/3

# Derivative at a point

*Just differentiate, then plug in the point.*

# Derivative at a point

Find $f'\left(\frac{\pi}{4}\right)$, where $f(x) = 20\cos x + 23\sin x$. [7]

[7] BMT 2023/1

# Derivative at a point

Find $f'\left(\frac{\pi}{4}\right)$, where $f(x) = 20\cos x + 23\sin x$. [7]

$$f'(x) = -20\sin x + 23\cos x$$

[7] BMT 2023/1

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$$f'(x) = -20\sin x + 23\cos x$$

$$f'\left(\tfrac{\pi}{4}\right) = 3 \cdot \tfrac{\sqrt{2}}{2} = \boxed{\frac{3\sqrt{2}}{2}}$$

[7] BMT 2023/1

# Area as an integral

*Area under a root curve is a power integral in disguise.*

# Area as an integral

Let $A$ be the area bounded by $x = 0$, $y = 0$, $x = 6$, and $y = \sqrt{kx}$ with $k > 0$. If $A = 36$, compute $k$. [8]

[8] BMT 2023/3

# Area as an integral

Let $A$ be the area bounded by $x = 0$, $y = 0$, $x = 6$, and $y = \sqrt{kx}$ with $k > 0$. If $A = 36$, compute $k$. [8]

$$A = \int_0^6 \sqrt{kx}\, dx = \sqrt{k}\left[\tfrac{2}{3}x^{3/2}\right]_0^6 = \sqrt{k} \cdot \tfrac{2}{3} \cdot 6^{3/2} = 4\sqrt{6} \cdot \sqrt{k} = 36$$

[8] BMT 2023/3

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$$\sqrt{k} = \frac{9}{\sqrt{6}}, \qquad k = \frac{81}{6} = \boxed{\frac{27}{2}}$$

[8] BMT 2023/3

# Related rates

*Relate $V$ to $s$, differentiate implicitly, then substitute.*

# Related rates

An ice cube stays a cube as it melts. Its volume decreases at a constant rate; with side 10 in it melts completely in 50 min. When the side is 4 in, at what rate (in/min) is the side decreasing? [9]

[9] BMT 2023/4

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$$\frac{dV}{dt} = -\frac{1000}{50} = -20, \qquad V = s^3 \;\Rightarrow\; 3s^2\frac{ds}{dt} = -20$$

[9] BMT 2023/4

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$$\frac{dV}{dt} = -\frac{1000}{50} = -20, \qquad V = s^3 \;\Rightarrow\; 3s^2\frac{ds}{dt} = -20$$

At $s = 4$:

$$\frac{ds}{dt} = -\frac{20}{48} = -\frac{5}{12} \quad \Rightarrow \quad \boxed{\frac{5}{12}\ \text{in/min}}$$

[9] BMT 2023/4

# Limit by factoring

*Factor the removable discontinuity, then evaluate.*

# Limit by factoring

Find $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$. [10]

[10] BMT 2024/1

# Limit by factoring

Find $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$. [10]

$$\frac{(x-2)(x^2+2x+4)}{x-2} = x^2 + 2x + 4 \to \boxed{12}$$

[10] BMT 2024/1

# Chain rule symmetry

*The chain rule's inner minus sign is the whole trick.*

# Chain rule symmetry

Given $f(x) = x^3 - 3x^2 + 3x - 1$ and $f(x) = g(44 - x)$, find $f'(20) + g'(24)$. [11]

[11] BMT 2024/2

# Chain rule symmetry

Given $f(x) = x^3 - 3x^2 + 3x - 1$ and $f(x) = g(44 - x)$, find $f'(20) + g'(24)$. [11]
Differentiating $f(x) = g(44 - x)$:

$$f'(x) = -g'(44 - x)$$

At $x = 20$, $g'(24) = -f'(20)$, so

$$\boxed{f'(20) + g'(24) = 0}$$

[11] BMT 2024/2

# Fixed point of differentiation

*Set $f$ equal to its derivative and solve the quadratic.*

# Fixed point of differentiation

Let $f(x) = x^2 - 11x + 31$. Distinct reals $a < b$ satisfy $f(a) = f'(a)$ and $f(b) = f'(b)$. Compute $10a + b$. [12]

[12] BMT 2025/1

# Fixed point of differentiation

Let $f(x) = x^2 - 11x + 31$. Distinct reals $a < b$ satisfy $f(a) = f'(a)$ and $f(b) = f'(b)$. Compute $10a + b$. [12]

$$x^2 - 11x + 31 = 2x - 11 \;\Rightarrow\; x^2 - 13x + 42 = (x-6)(x-7) = 0$$

$$\{a, b\} = \{6, 7\}, \qquad \boxed{10a + b = 67}$$

[12] BMT 2025/1

# Limit of a derivative at infinity

*Only the leading terms of numerator and root survive at infinity.*

# Limit of a derivative at infinity

Let $f(x) = \sqrt{3x^2 + 2x + 1}$. Compute $\lim_{x \to \infty} f'(x)$. [13]

[13] BMT 2025/3

# Limit of a derivative at infinity

Let $f(x) = \sqrt{3x^2 + 2x + 1}$. Compute $\lim_{x \to \infty} f'(x)$. [13]

$$f'(x) = \frac{6x + 2}{2\sqrt{3x^2 + 2x + 1}} \to \frac{6}{2\sqrt{3}} = \boxed{\sqrt{3}}$$

[13] BMT 2025/3

# The $1^\infty$ limit form

*Take logs; L'Hôpital turns* $1^\infty$ *into a finite limit.*

# The $1^\infty$ limit form

Compute $\lim_{n\to\infty} \left(\frac{2\arctan n}{\pi}\right)^n$. [14]

[14] BMT 2025/4

# The $1^\infty$ limit form

Compute $\lim_{n\to\infty}\left(\frac{2\arctan n}{\pi}\right)^n$. [14] Let $L$ be the limit:

$$\ln L = \lim n \cdot \ln \frac{2\arctan n}{\pi} = \lim \frac{\ln(2\arctan n/\pi)}{1/n},$$

a $0/0$ form.

[14] BMT 2025/4

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a $0/0$ form. By L'Hôpital:

$$\frac{(2/\pi)(1/(1+n^2))/(2\arctan n/\pi)}{-1/n^2} \to -\frac{2}{\pi}$$

[14] BMT 2025/4

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a $0/0$ form. By L'Hôpital:

$$\frac{(2/\pi)(1/(1+n^2))/(2\arctan n/\pi)}{-1/n^2}\to -\frac{2}{\pi}$$

$$\boxed{L = e^{-2/\pi}}$$

[14] BMT 2025/4