Easy Problems

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Fourteen one-step warm-ups. Each takes a single idea — good for a quick pass before the harder handouts.

Problem 1 (BMT 2020/1)**.** Let $f(x) = \frac{x^{2020}}{2020} + 2020!$. Compute $f''(1)$.

Solution. $f'(x) = x^{2019}$ and $f''(x) = 2019x^{2018}$, so $f''(1) = 2019$. □

Problem 2 (BMT 2020/2)**.** Compute $\int_{-20}^{20}(20 - |x|)\,dx$.

Solution. The integrand is even: $2\int_0^{20}(20 - x)\,dx = 2\left[20x - \frac{x^2}{2}\right]_0^{20} = 2(400 - 200) = 400$. □

Problem 3 (BMT 2021/1)**.** Let $g(x) = \int_{2021}^{x}(e^t - 2t)\,dt$. Compute $g'(2021)$.

Solution. By FTC, $g'(x) = e^x - 2x$, so $g'(2021) = e^{2021} - 4042$. □

Problem 4 (BMT 2021/2)**.** Let $f(x) = (x+3)(2x+5)(3x+7)(x+1)$. Compute $f''''(5)$.

Solution. f is quartic with leading coefficient $1 \cdot 2 \cdot 3 \cdot 1 = 6$, so $f''''(x) = 6 \cdot 4! = 144$ is constant. □

Problem 5 (BMT 2022/1)**.** Compute $\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 x$.

Solution. $2\sin x\cos x - 2\sin x\cos x = 0$. □

Problem 6 (BMT 2022/3)**.** Compute $\int_0^1 e^x \cdot e^{e^x} \cdot e^{e^{e^x}}\,dx$.

Solution. By the chain rule, $e^x \cdot e^{e^x} = \left(e^{e^x}\right)'$, and multiplying by $e^{e^{e^x}}$ gives $\left(e^{e^{e^x}}\right)' = e^{e^{e^x}} \cdot \left(e^{e^x}\right)'$. So the integrand is $\frac{d}{dx}e^{e^{e^x}}$, and the integral is $\left[e^{e^{e^x}}\right]_0^1 = e^{e^e} - e^e$. □

Problem 7 (BMT 2023/1)**.** Find $f'\left(\frac{\pi}{4}\right)$, where $f(x) = 20\cos x + 23\sin x$.

Solution. $f'(x) = -20\sin x + 23\cos x$, so $f'\left(\frac{\pi}{4}\right) = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$. □

Problem 8 (BMT 2023/3)**.** Let A be the area bounded by $x = 0$, $y = 0$, $x = 6$, and $y = \sqrt{kx}$ with $k > 0$. If $A = 36$, compute k.

Solution. $A = \int_0^6 \sqrt{kx}\,dx = \sqrt{k} \cdot \frac{2}{3} \cdot 6^{3/2} = 4\sqrt{6} \cdot \sqrt{k} = 36$, so $\sqrt{k} = \frac{9}{\sqrt{6}}$ and $k = \frac{27}{2}$. □

Problem 9 (BMT 2023/4)**.** An ice cube stays a cube as it melts. Its volume decreases at a constant rate; with side 10 in it melts completely in 50 min. When the side is 4 in, at what rate (in/min) is the side decreasing?

Solution. $\frac{dV}{dt} = -\frac{1000}{50} = -20$. From $V = s^3$, $3s^2\frac{ds}{dt} = -20$; at $s = 4$, $\frac{ds}{dt} = -\frac{20}{48} = -\frac{5}{12}$, so the side decreases at $\frac{5}{12}$ in/min. □

Problem 10 (BMT 2024/1)**.** Find $\lim_{x\to 2} \frac{x^3-8}{x-2}$.

Solution. Factor: $\frac{(x-2)(x^2+2x+4)}{x-2} = x^2 + 2x + 4 \to 12$. □

Problem 11 (BMT 2024/2)**.** Given $f(x) = x^3 - 3x^2 + 3x - 1$ and $f(x) = g(44 - x)$, find $f'(20) + g'(24)$.

Solution. Differentiating $f(x) = g(44 - x)$ gives $f'(x) = -g'(44 - x)$. At $x = 20$, $g'(24) = -f'(20)$, so the sum is 0. □

Problem 12 (BMT 2025/1)**.** Let $f(x) = x^2 - 11x + 31$. Distinct reals $a < b$ satisfy $f(a) = f'(a)$ and $f(b) = f'(b)$. Compute $10a + b$.

Solution. $x^2 - 11x + 31 = 2x - 11$ gives $x^2 - 13x + 42 = 0$, so $\{a, b\} = \{6, 7\}$ and $10a + b = 67$. □

Problem 13 (BMT 2025/3)**.** Let $f(x) = \sqrt{3x^2 + 2x + 1}$. Compute $\lim_{x\to\infty} f'(x)$.

Solution. $f'(x) = \frac{6x+2}{2\sqrt{3x^2+2x+1}} \to \frac{6}{2\sqrt{3}} = \sqrt{3}$. □

Problem 14 (BMT 2025/4)**.** Compute $\lim_{n\to\infty} \left(\frac{2\arctan n}{\pi}\right)^n$.

Solution. Take logs: $n \ln \frac{2\arctan n}{\pi} \to -\frac{2}{\pi}$ by L'Hôpital, so the limit is $e^{-2/\pi}$. □

Answers: 2019; 400; $e^{2021} - 4042$; 144; 0; $e^{e^e} - e^e$; $\frac{3\sqrt{2}}{2}$; $\frac{27}{2}$; $\frac{5}{12}$; 12; 0; 67; $\sqrt{3}$; $e^{-2/\pi}$.