Integral Symmetry and Pairing

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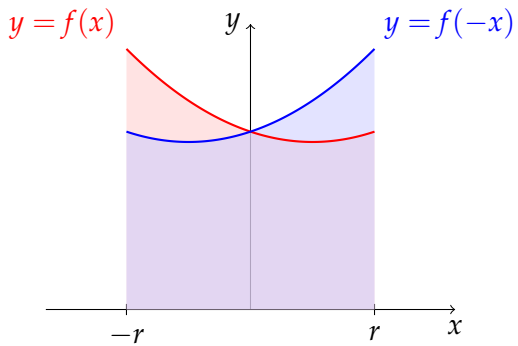
$$\int_{-r}^{r} f(x)\,dx = \int_{-r}^{r} f(-x)\,dx$$



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Pairing via $x \mapsto -x$ or $x \mapsto a + b - x$

Add the integral to its mirror image — the awkward factors cancel, the two copies add to something easy.

Pairing x and $-x$: the move

Pair the integral with its mirror image — the hard parts cancel, and the two copies add to something easy.

On a symmetric interval, substitute $x \mapsto -x$ (on $[a, b]$ use $x \mapsto a + b - x$) and **add the two copies**:

$$I = \int_{-r}^{r} f(x)\,dx \quad \text{and} \quad I = \int_{-r}^{r} f(-x)\,dx \implies 2I = \int_{-r}^{r} \big(f(x) + f(-x)\big)\,dx.$$

Works when $f(x) + f(-x)$ is much simpler than $f(x)$ — e.g. $\frac{1}{1+e^{x}} + \frac{1}{1+e^{-x}} = 1$.

Example: BMT 2018/5 (statement)

What is

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{-x}}\,dx?$$

Example: BMT 2018/5 (solution)

Let I be the integral. Substitute $x \mapsto -x$:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx = \int_{-\pi/2}^{\pi/2} \frac{e^{-x}\cos x}{1+e^{-x}}\,dx,$$

since $\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}$.

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since $\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}$.

Why rewrite? With the common denominator $1+e^{-x}$ the two copies add to $\cos x$ times 1. Adding,

$$2I = \int_{-\pi/2}^{\pi/2} \cos x \left(\frac{1}{1+e^{-x}} + \frac{e^{-x}}{1+e^{-x}}\right) dx = \int_{-\pi/2}^{\pi/2} \cos x\,dx = 2.$$

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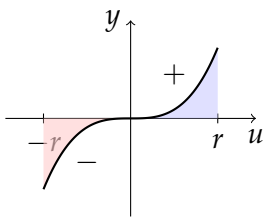
Hence $\boxed{I = 1}$.

Centering the interval, killing the odd part

Shift $u = x - a$ to center the interval — odd powers die, even powers double.

Center the interval and kill the odd part: the move

Powers of $(x - a)$? Shift $u = x - a$ to center at 0.



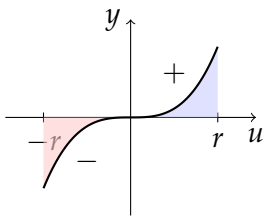
odd lobes cancel

Center the interval and kill the odd part: the move

Powers of $(x-a)$? Shift $u = x - a$ to center at 0.
On $[-r, r]$:

$$\int_{-r}^{r} u^{2k+1}\,du = 0, \qquad \int_{-r}^{r} u^{2k}\,du = 2\int_{0}^{r} u^{2k}\,du,$$

so every **odd power dies** and every **even power doubles**.



odd lobes cancel

Example: BMT 2023/2 (statement)

Compute

$$\int_0^4 \left[(x-2)^5 + (x-2)^6 + (x-2)^7\right] dx.$$

Example: BMT 2023/2 (solution)

Center the interval first: with $u = x - 2$,

$$\int_{-2}^{2} \left(u^5 + u^6 + u^7\right) du.$$

Example: BMT 2023/2 (solution)

Center the interval first: with $u = x - 2$,

$$\int_{-2}^{2} \left(u^5 + u^6 + u^7\right) du.$$

Why center? On $[-2, 2]$ the odd powers die, leaving only the even power:

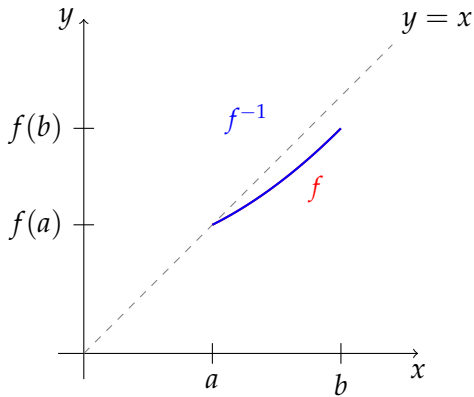
$$\int_{-2}^{2} u^6 \, du = 2 \int_{0}^{2} u^6 \, du = 2 \cdot \frac{128}{7} = \boxed{\frac{256}{7}}.$$

The inverse-area identity

Reflect the graph across $y = x$ — the two areas tile $bf(b) - af(a)$.

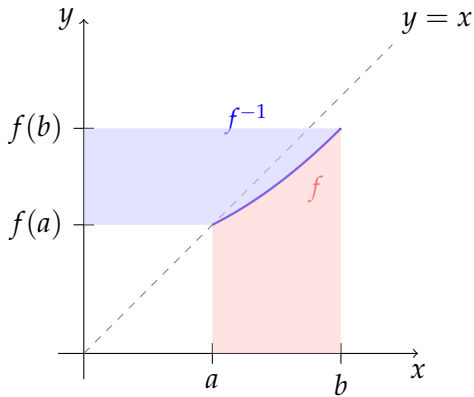
The inverse-area identity: the move

For f continuous and strictly increasing, reflect the graph across $y = x$.



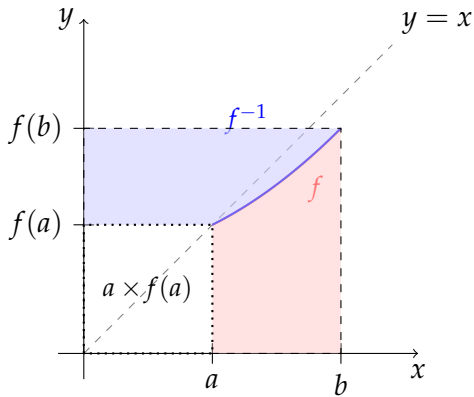
The inverse-area identity: the move

For f continuous and strictly increasing, reflect the graph across $y = x$. The area under f plus the area under f^{-1}



The inverse-area identity: the move

For f continuous and strictly increasing, reflect the graph across $y = x$. The area under f plus the area under f^{-1} tiles the $b \times f(b)$ rectangle minus the $a \times f(a)$ corner, so

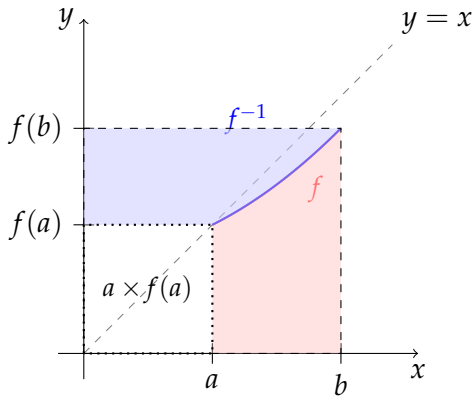


The inverse-area identity: the move

For f continuous and strictly increasing, reflect the graph across $y = x$. The area under f plus the area under f^{-1} tiles the $b \times f(b)$ rectangle minus the $a \times f(a)$ corner, so

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

Lets you integrate an inverse function **without ever finding it**.



Example: BMT 2021/7 (statement)

Let $c(x) = e^x + e^{-2x}$, defined on the interval $1 \leq x \leq 2$. Let $c^{-1}(x)$ be the inverse of $c(x)$. Compute

$$\int_{c(1)}^{c(2)} c^{-1}(x)\, dx.$$

Example: BMT 2021/7 (try it)

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c^{-1} has no closed form — what area picture relates $\int c^{-1}$ to $\int c$?

Example: BMT 2021/7 (solution)

Apply the inverse-area identity to $f = c$ on $[1, 2]$. Since $c'(x) = e^x - 2e^{-2x} > 0$ on $[1, 2]$ ($e^{3x} > 2$ there), c is strictly increasing, so c^{-1} exists and the identity applies:

$$\int_{c(1)}^{c(2)} c^{-1}(x)\,dx = 2c(2) - c(1) - \int_1^2 c(x)\,dx.$$

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Now $c(1) = e + e^{-2}$ and $c(2) = e^2 + e^{-4}$, and

$$\int_1^2 (e^x + e^{-2x})\,dx = \left[e^x - \tfrac{1}{2}e^{-2x}\right]_1^2 = \left(e^2 - \tfrac{1}{2}e^{-4}\right) - \left(e - \tfrac{1}{2}e^{-2}\right).$$

Example: BMT 2021/7 (solution, continued)

So far: $\int c^{-1} = 2c(2) - c(1) - \int_1^2 c(x)\,dx$ — now substitute the values.

Subtracting,

$$\left(2e^2 + 2e^{-4} - e - e^{-2}\right) - \left(e^2 - e - \tfrac{1}{2}e^{-4} + \tfrac{1}{2}e^{-2}\right) = \boxed{e^2 + \tfrac{5}{2}e^{-4} - \tfrac{3}{2}e^{-2}}.$$

Practice

Your turn: the same pairing move as the first example.

Practice: pair the copies

Substitute $x \mapsto -x$ and add the two copies, as in the first example.

Compute

$$\int_{-1}^{1} \frac{x^2}{1+e^x}\, dx.$$

Practice: answer

$$\boxed{\frac{1}{3}}$$