Integral Symmetry and Pairing

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Pair the integral with its mirror image — the hard parts cancel, and the two copies add to something easy.

Move 1 (pairing): on a symmetric interval, substitute $x \mapsto -x$ (or $x \mapsto a+b-x$ on $[a,b]$) and add the two copies; the awkward factors cancel.

Move 2 (center and kill the odd part): if the integrand is built from powers of $(x-a)$, shift $u = x-a$ to center the interval at 0. On a symmetric interval $[-r,r]$,

$$\int_{-r}^{r} u^{2k+1}\,du = 0, \qquad \int_{-r}^{r} u^{2k}\,du = 2\int_{0}^{r} u^{2k}\,du,$$

so every odd power dies and every even power doubles.

Move 3 (inverse-area identity): for a continuous, strictly increasing f, reflect the graph across $y = x$: the area under f plus the area under f^{-1} tiles the $b \times f(b)$ rectangle minus the $a \times f(a)$ corner, so

$$\int_{a}^{b} f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

Problem 1 (BMT 2018/5)**.** What is

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{-x}}\,dx?$$

Solution. Let I be the integral. Substitute $x \mapsto -x$:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{x}}\,dx = \int_{-\pi/2}^{\pi/2} \frac{e^{-x}\cos x}{1+e^{-x}}\,dx,$$

since $\frac{1}{1+e^{x}} = \frac{e^{-x}}{1+e^{-x}}$. Adding the two copies,

$$2I = \int_{-\pi/2}^{\pi/2} \cos x \left(\frac{1}{1+e^{-x}} + \frac{e^{-x}}{1+e^{-x}}\right) dx = \int_{-\pi/2}^{\pi/2} \cos x\,dx = 2.$$

Hence $I = 1$. □

Problem 2 (BMT 2023/2)**.** Compute

$$\int_{0}^{4} \left[(x-2)^5 + (x-2)^6 + (x-2)^7\right] dx.$$

Solution. Center the interval first: with $u = x - 2$,

$$\int_{-2}^{2} \left(u^5 + u^6 + u^7\right) du.$$

By Move 2 the odd powers die, leaving

$$\int_{-2}^{2} u^6\, du = 2\int_{0}^{2} u^6\, du = 2 \cdot \frac{128}{7} = \frac{256}{7}.$$

□

Problem 3 (BMT 2021/7)**.** Let $c(x) = e^x + e^{-2x}$, defined on the interval $1 \le x \le 2$. Let $c^{-1}(x)$ be the inverse of $c(x)$. Compute

$$\int_{c(1)}^{c(2)} c^{-1}(x)\, dx.$$

Solution. Since $c'(x) = e^x - 2e^{-2x} > 0$ on $[1,2]$ (because $e^{3x} \ge e^3 > 2$ there), c is continuous and strictly increasing, so c^{-1} exists and Move 3 applies to $f = c$ on $[1,2]$:

$$\int_{c(1)}^{c(2)} c^{-1}(x)\, dx = 2c(2) - c(1) - \int_{1}^{2} c(x)\, dx.$$

Now $c(1) = e + e^{-2}$, $c(2) = e^2 + e^{-4}$, and

$$\int_{1}^{2} (e^x + e^{-2x})\, dx = \left[e^x - \tfrac{1}{2}e^{-2x}\right]_1^2 = \left(e^2 - \tfrac{1}{2}e^{-4}\right) - \left(e - \tfrac{1}{2}e^{-2}\right).$$

Subtracting,

$$\left(2e^2 + 2e^{-4} - e - e^{-2}\right) - \left(e^2 - e - \tfrac{1}{2}e^{-4} + \tfrac{1}{2}e^{-2}\right) = e^2 + \tfrac{5}{2}e^{-4} - \tfrac{3}{2}e^{-2}.$$

□

Exercise 4. Compute $\int_{-1}^{1} \frac{x^2}{1+e^x}\, dx$. Hint: 1

Answers: 1; 256/7; $e^2 + \frac{5}{2}e^{-4} - \frac{3}{2}e^{-2}$; 1/3.

Hints

1. Substitute $x \mapsto -x$ and add the two copies, as in the first example.