Limits of Integrals

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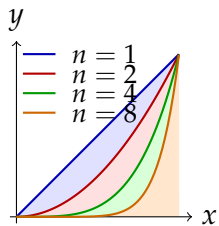

Key idea: the area under $y = x^n$ on $[0, 1]$ dies as $n \to \infty$:

$$\int_0^1 x^n \, dx = \frac{1}{n+1} \to 0.$$

The “tail dies” move

A limit wrapped around an integral surrenders to integration by parts: push the n-dependence onto a dying factor, bound the rest, and read off the boundary term.

The tail dies: why x^{n+1} vanishes

A limit wrapped around an integral surrenders to integration by parts.

In $\lim_{n\to\infty} \int_0^1 (\text{something with } n)\, dx$:

- Isolate the n-dependent factor — often x^{n+1}.
- On $[0, 1)$ it collapses to 0 as $n \to \infty$; it dies *everywhere except* at the endpoint $x = 1$.
- If everything else stays bounded, the whole tail integral dies by a squeeze.

The move

Push all the n-dependence onto a dying factor like x^{n+1}, bound the rest, and read off the boundary term.

The squeeze bound: only the boundary survives

If $|g(x)| \le M$ on $[0,1]$:

$$\left| \int_0^1 x^{n+1} g(x)\, dx \right| \le M \int_0^1 x^{n+1}\, dx = \frac{M}{n+2} \longrightarrow 0.$$

So after integration by parts

$$\int_0^1 u\, dv = \underbrace{\Big[\text{boundary term}\Big]_0^1}_{\text{the only survivor}} - \underbrace{\int_0^1 x^{n+1}(\text{bounded})\, dx}_{\text{dies by the squeeze}},$$

only the $x = 1$ boundary term can survive.

The “tail dies” move: Example

Compute

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n}{\sqrt{4x^3 - x + 1}}\, dx.$$

Integrate by parts to push the n-dependence onto a dying x^{n+1} factor. [1]

[1] BMT 2018/4

The “tail dies” move: Solution

Why parts? The loose factor n is the trouble; integrating nx^n gives $\frac{n}{n+1}x^{n+1}$, which is bounded. Take $u = (4x^3 - x + 1)^{-1/2}$ and $dv = nx^n\,dx$:

$$v = \frac{nx^{n+1}}{n+1}, \qquad du = -\frac{12x^2 - 1}{2(4x^3 - x + 1)^{3/2}}\,dx,$$

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so

$$\int_0^1 u\,dv = \underbrace{\left[\frac{nx^{n+1}}{(n+1)\sqrt{4x^3 - x + 1}}\right]_0^1}_{\text{boundary}} + \underbrace{\int_0^1 \frac{nx^{n+1}}{n+1} \cdot \frac{12x^2 - 1}{2(4x^3 - x + 1)^{3/2}}\,dx}_{\text{tail dies}}.$$

The “tail dies” move: Solution (cont.)

So far: $\int u\,dv = \text{boundary} + \text{tail}$.

Boundary term: at $x = 1$ the root is $\sqrt{4} = 2$, at $x = 0$ the term is 0, so it equals $\frac{n}{(n+1)\cdot 2} \to \frac{1}{2}$.

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Tail: $\frac{n}{n+1} \le 1$, and the cofactor $\frac{12x^2-1}{2(4x^3-x+1)^{3/2}}$ is bounded on $[0,1]$ because the denominator never gets near 0: the minimum of $4x^3 - x + 1$ (at $x = \frac{1}{2\sqrt{3}}$) is $1 - \frac{1}{3\sqrt{3}} > 0.8$. So the tail vanishes by the squeeze.

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The limit is the surviving boundary term: $\boxed{\frac{1}{2}}$

FTC plus L'Hôpital on a parameter

When the limit looks like $\frac{0}{0}$ around an integral, rewrite with a small parameter and hit it with FTC plus L'Hôpital.

FTC plus L'Hôpital on a parameter

A limit wrapped around an integral surrenders to FTC plus L'Hôpital on a parameter.

Recipe:

- Set $h = \frac{1}{n}$ so $n \to \infty$ becomes $h \to 0^+$; the integral becomes $\int_0^h (\cdots)\, dx \to 0$; divided by a power of h it is a $\frac{0}{0}$ form.
- FTC: $\dfrac{d}{dh} \displaystyle\int_0^h f = f(h)$.
- L'Hôpital once (or twice) kills the parameter; standard limits like $h^h \to 1$ finish it.

FTC plus L'Hôpital on a parameter: Example

The limit

$$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1}\,dx$$

can be written in the form $\frac{p}{q}$*, where* p *and* q *are relatively prime positive integers. Compute* $p+q$. Set $h = \frac{1}{n}$; this is $\frac{0}{0}$ in disguise. [2]

[2] BMT 2020/7

FTC plus L'Hôpital on a parameter: Solution

With $h = \frac{1}{n}$:

$$\lim_{h \to 0^+} \frac{\int_0^h x^{x+1}\, dx}{h^2},$$

a $\frac{0}{0}$ form.

FTC plus L'Hôpital on a parameter: Solution

With $h = \frac{1}{n}$:

$$\lim_{h \to 0^+} \frac{\int_0^h x^{x+1}\,dx}{h^2},$$

a $\frac{0}{0}$ form. Why L'Hôpital? Top and bottom both tend to 0, and FTC differentiates the top for free: $\frac{d}{dh}\int_0^h x^{x+1}\,dx = h^{h+1}$.

$$\lim_{h \to 0^+} \frac{\int_0^h x^{x+1}\,dx}{h^2} = \lim_{h \to 0^+} \frac{h^{h+1}}{2h} = \frac{1}{2}\lim_{h \to 0^+} h^h = \frac{1}{2},$$

since $h^h = e^{h \ln h}$ and $h \ln h \to 0$. So $p = 1$, $q = 2$, and $\boxed{p + q = 3}$.

The 1^∞ move: take logarithms

A 1^∞ form surrenders to logarithms: the exponent lands on the bottom, and FTC plus L'Hôpital take it from there.

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Recipe:

- If $L = \lim(1+f(x))^{1/x}$, a 1^∞ form, take logarithms:

$$\ln L = \lim_{x\to 0} \frac{\ln(1+f(x))}{x},$$

  now a $\frac{0}{0}$ form.
- L'Hôpital: $\ln L = \lim \dfrac{f'(x)}{1+f(x)}$, where FTC supplies $f'(x)$.
- Watch the integrand at $t = 0$: it may need a continuous extension (here $\frac{\cos t - 1}{t^2} \to -\frac{1}{2}$).

The 1^∞ move: Example

Compute

$$\lim_{x\to 0}\left(1+\int_0^x \frac{\cos t-1}{t^2}\,dt\right)^{1/x}.$$

This is 1^∞: take logarithms, then FTC plus L'Hôpital. [3]

[3] BMT 2022/7

The 1^∞ move: Solution

Set $f(x) = \int_0^x \frac{\cos t - 1}{t^2}\, dt$; the integrand extends continuously at $t = 0$ with value $-\frac{1}{2}$, so $f(0) = 0$ and by FTC

$$f'(x) = \frac{\cos x - 1}{x^2} \longrightarrow -\frac{1}{2} \quad (x \to 0).$$

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Take logarithms:

$$\ln L = \lim_{x \to 0} \frac{\ln(1 + f(x))}{x},$$

a $\frac{0}{0}$ form. By L'Hôpital:

$$\ln L = \lim_{x \to 0} \frac{f'(x)}{1 + f(x)} = \frac{-1/2}{1} = -\frac{1}{2},$$

so $\boxed{L = e^{-1/2} = \frac{1}{\sqrt{e}}}$.

Practice exercises

Your turn — try it with the moves above.

Practice

Compute

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n}{1+x}\,dx.$$

Integrate by parts to expose the boundary term; the tail with x^{n+1} dies. [4]

[4] BMT 2018/4 variant

Practice: answers

Answers to the handout's problems, in order:

- BMT 2018/4: $\frac{1}{2}$
- BMT 2020/7: 3
- BMT 2022/7: $\frac{1}{\sqrt{e}}$
- Practice exercise: $\frac{1}{2}$