# Limits of Integrals

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A limit wrapped around an integral surrenders to integration by parts or to FTC plus L'Hôpital on a parameter.

**The "tail dies" move.** In $\lim_{n\to\infty} \int_0^1 (\text{something with } n)\, dx$, isolate the $n$-dependent factor — often $x^{n+1}$. On $[0,1)$ it collapses to 0 as $n \to \infty$; if everything else stays bounded on the interval, the whole tail integral dies by a squeeze. Only a boundary term can survive, because the factor dies everywhere *except* at the endpoint $x = 1$. So the game is: use parts to push all the $n$-dependence onto a dying factor like $x^{n+1}$, bound the rest, and read off the boundary term.

**Problem 1** (BMT 2018/4)**.** Compute

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n}{\sqrt{4x^3 - x + 1}}\, dx.$$

*Solution.* Integrate by parts to move the $n$ onto a harmless factor. With $u = (4x^3 - x + 1)^{-1/2}$ and $dv = nx^n\, dx$,

$$v = \frac{nx^{n+1}}{n+1}, \qquad du = -\frac{12x^2 - 1}{2(4x^3 - x + 1)^{3/2}}\, dx,$$

so

$$\int_0^1 u\, dv = \left[\frac{nx^{n+1}}{(n+1)\sqrt{4x^3 - x + 1}}\right]_0^1 + \int_0^1 \frac{nx^{n+1}}{n+1} \cdot \frac{12x^2 - 1}{2(4x^3 - x + 1)^{3/2}}\, dx.$$

The boundary term is $\frac{n}{(n+1)\cdot 2} \to \frac{1}{2}$. By the tail-dies move, the remaining integral vanishes: the $x^{n+1}$ factor dies on $[0,1)$ while the other factors stay bounded. Hence the limit is $\frac{1}{2}$. □

**Problem 2** (BMT 2020/7)**.** The limit

$$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1}\, dx$$

can be written in the form $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Compute $p + q$.

*Solution.* Set $h = \frac{1}{n}$, so the limit becomes

$$\lim_{h\to 0^+} \frac{\int_0^h x^{x+1}\, dx}{h^2},$$

a $\frac{0}{0}$ form. By FTC and L'Hôpital,

$$\lim_{h\to 0^+} \frac{\int_0^h x^{x+1}\,dx}{h^2} = \lim_{h\to 0^+} \frac{h^{h+1}}{2h} = \frac{1}{2}\lim_{h\to 0^+} h^h = \frac{1}{2},$$

since $h^h \to 1$. Thus $p = 1$, $q = 2$, and $p + q = 3$. □

**Problem 3** (BMT 2022/7)**.** Compute

$$\lim_{x\to 0}\left(1 + \int_0^x \frac{\cos t - 1}{t^2}\,dt\right)^{1/x}.$$

*Solution.* This is a $1^\infty$ form. Set $f(x) = \int_0^x \frac{\cos t - 1}{t^2}\,dt$; the integrand extends continuously at $t = 0$ with value $-\frac{1}{2}$, so $f$ is differentiable with $f(0) = 0$ and, by FTC, $f'(x) = \frac{\cos x - 1}{x^2}$, which tends to $-\frac{1}{2}$ as $x \to 0$. Take logarithms:

$$\ln L = \lim_{x\to 0} \frac{\ln(1 + f(x))}{x},$$

a $\frac{0}{0}$ form. By L'Hôpital,

$$\ln L = \lim_{x\to 0} \frac{f'(x)}{1 + f(x)} = \frac{-1/2}{1} = -\frac{1}{2},$$

so $L = e^{-1/2} = \frac{1}{\sqrt{e}}$. □

**Exercise 4** (BMT 2018/4 variant)**.** Compute $\lim_{n\to\infty} \int_0^1 \frac{nx^n}{1+x}\,dx$. Hint: 1

*Answers:* $1/2$; $3$; $1/\sqrt{e}$; $1/2$.

## Hints

1. Integrate by parts to expose the boundary term; the tail with $x^{n+1}$ dies.