Series–Integral Conversion

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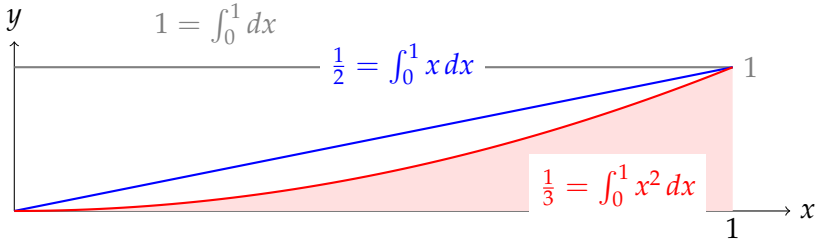

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The whole technique: $\frac{1}{m}$ *is the area under* x^{m-1} *on* $[0, 1]$.

The Conversion Identity

Replace $\frac{1}{m}$ with $\int_0^1 x^{m-1}\,dx$ to turn a sum into an integral of a geometric series you can sum in closed form.

The Conversion Identity: The Move

Replace $\frac{1}{m}$ with $\int_0^1 x^{m-1}\,dx$ to turn a sum into an integral of a geometric series you can sum in closed form.

The whole technique rests on one identity:

$$\frac{1}{m} = \int_0^1 x^{m-1}\,dx \qquad (m \geq 1).$$

Whenever a series has a $\frac{1}{m}$-type factor, convert it:

$$\sum(\text{rest}) \cdot \frac{1}{m} = \int_0^1 \sum(\text{rest}) \cdot x^{m-1}\,dx,$$

and the inner sum is usually a geometric series that collapses in closed form.

The Conversion Identity: The three-step recipe

The three-step recipe

1. Spot the $\frac{1}{m}$ factor and convert it to an integral.
2. Swap sum and integral, then sum the geometric series.
3. Evaluate the remaining integral.

Two Series Facts, One Factorization

Two recurring series facts ($-\ln(1-x)$ and the Leibniz series) plus the double-sum factorization move.

Series fact: the $-\ln(1-x)$ series

Differentiating $-\ln(1-x)$ gives $\frac{1}{1-x} = \sum_{a\geq 0} x^a$; integrating termwise,

$$\int_0^x \frac{dt}{1-t} = \sum_{a\geq 1} \frac{x^a}{a},$$

so

$$-\ln(1-x) = \sum_{a\geq 1} \frac{x^a}{a} \qquad (|x| < 1).$$

Series fact: the Leibniz series

The Leibniz series — derived, not memorized:

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots = \tfrac{\pi}{4}.$$

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Step 2: $\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots$ is geometric; integrate termwise:

$$\arctan x = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \cdots$$

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Step 3: at $x = 1$,

$$\boxed{1 - \tfrac{1}{3} + \tfrac{1}{5} - \cdots = \arctan 1 = \tfrac{\pi}{4}}.$$

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Why is $x = 1$ allowed, when the ratio $-t^2$ reaches size 1? Stop after N terms: the leftover $\frac{(-t^2)^N}{1+t^2}$ integrates to at most $\int_0^1 t^{2N}\,dt = \frac{1}{2N+1} \to 0$.

The double-sum factorization move

One more move: a double sum whose terms factor as $A_a \cdot B_b$ splits into a product,

$$\sum_{a,b} A_a B_b = \left(\sum_a A_a\right)\left(\sum_b B_b\right),$$

which is how the two $-\ln(1-x)$ series will appear in the first example.

For instance $\displaystyle\sum_{a,b\geq 1} \frac{x^a}{a}\cdot\frac{x^b}{b} = \left(\sum_{a\geq 1}\frac{x^a}{a}\right)^2 = \ln^2(1-x).$

Example: BMT 2020/8

BMT 2020/8: factor the denominator, convert $1/(a+b+2)$ *to an integral, factor the double sum.*

The Conversion Identity: Example

The summation

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{a^2b+2ab+ab^2}$$

can be written in the form $\frac{m}{n}$*, where* m *and* n *are relatively prime positive integers. Compute* $m+n$*.* [1]

[1] BMT 2020/8

The Conversion Identity: Solution (1/2)

The denominator factors: $a^2b + 2ab + ab^2 = ab(a + b + 2)$.

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$$\frac{1}{ab(a+b+2)} = \frac{1}{ab} \int_0^1 x^{a+b+1}\, dx.$$

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$$\frac{1}{ab(a+b+2)} = \frac{1}{ab} \int_0^1 x^{a+b+1}\, dx.$$

Step 2 — swap sum and integral (all terms positive): since $\frac{x^{a+b+1}}{ab} = x \cdot \frac{x^a}{a} \cdot \frac{x^b}{b}$, the double sum factors:

$$\sum_{a,b\geq 1} \frac{1}{ab(a+b+2)} = \int_0^1 x \left(\sum_{a\geq 1} \frac{x^a}{a} \right) \left(\sum_{b\geq 1} \frac{x^b}{b} \right) dx.$$

The Conversion Identity: Solution (2/2)

Each inner sum is $-\ln(1-x)$, so the sum is $\displaystyle\int_0^1 x \ln^2(1-x)\,dx$.

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Step 3 — evaluate: with $u = 1 - x$,

$$\int_0^1 (1-u)\ln^2 u\,du = \int_0^1 \ln^2 u\,du - \int_0^1 u\ln^2 u\,du.$$

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Both integrals by parts:

$$\int_0^1 \ln^2 u\,du = [u\ln^2 u]_0^1 - 2\int_0^1 \ln u\,du = 2$$

$$\int_0^1 u\ln^2 u\,du = [u^2\ln^2 u/2]_0^1 - \int_0^1 u\ln u\,du = 1/4.$$

So

$$2 - \frac{1}{4} = \frac{7}{4}.$$

Thus $m = 7$, $n = 4$, and $m+n = \boxed{11}$.

Example: BMT 2025/5

BMT 2025/5: slice the integral at the integers so it becomes the Leibniz series.

The Conversion Identity: Example

For each real number n, let $\lfloor n \rfloor$ denote the greatest integer less than or equal to n, and let $\lceil n \rceil$ denote the least integer greater than or equal to n. Compute

$$\int_0^\infty \frac{(-1)^{\lfloor x \rfloor}}{\lceil x \rceil^2 - \lfloor x \rfloor^2}\, dx.$$

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[2] BMT 2025/5

The Conversion Identity: Solution

This one runs the conversion in reverse: slicing the integral at the integers turns it into a sum.

On each open interval $(i, i+1)$ the floors and ceilings are constant: $\lfloor x \rfloor = i$, $\lceil x \rceil = i+1$ (single points do not change an integral). Since $(i+1)^2 - i^2 = 2i+1$,

$$\int_0^\infty \cdots = \sum_{i=0}^{\infty} \int_i^{i+1} \frac{(-1)^i}{2i+1}\, dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1}.$$

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This is the Leibniz series derived above, so the value is

$$\boxed{\frac{\pi}{4}}.$$

Example: BMT 2024/8

BMT 2024/8: partial fractions first, then turn each $1/k$ *into an integral and sum the ratio* $-x^3$ *geometric series.*

The Conversion Identity: Example

Evaluate

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{9n^2 - 12n + 2}{(3n)(3n-1)(3n-2)}.$$

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[3] BMT 2024/8

The Conversion Identity: Try it

Evaluate

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{9n^2 - 12n + 2}{(3n)(3n-1)(3n-2)}.$$

The summand splits over $3n-2$, $3n-1$, $3n$—what does partial fractions give? [4]

[4] BMT 2024/8

The Conversion Identity: Solution (1/2)

Partial fractions give

$$\frac{9n^2 - 12n + 2}{(3n)(3n-1)(3n-2)} = -\frac{1}{3n-2} + \frac{1}{3n-1} + \frac{1}{3n},$$

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so the series expands to

$$-1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots.$$

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Write each $\frac{1}{k}$ as $\int_0^1 x^{k-1}\,dx$ and regroup by powers of x:

$$\int_0^1 \left[(-1 + x + x^2) + (x^3 - x^4 - x^5) + (-x^6 + x^7 + x^8) + \cdots\right] dx.$$

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Partial fractions give

$$\frac{9n^2-12n+2}{(3n)(3n-1)(3n-2)} = -\frac{1}{3n-2}+\frac{1}{3n-1}+\frac{1}{3n},$$

so the series expands to

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Write each $\frac{1}{k}$ as $\int_0^1 x^{k-1}\,dx$ and regroup by powers of x:

$$\int_0^1 \left[(-1+x+x^2)+(x^3-x^4-x^5)+(-x^6+x^7+x^8)+\cdots\right]dx.$$

The bracketed terms form a geometric series: the first term is $-1+x+x^2$ and the ratio is $-x^3$, so they sum to $\frac{-1+x+x^2}{1+x^3}$.

The Conversion Identity: Solution (2/2)

Since, using $1 + x^3 = (1 + x)(1 - x + x^2)$,

$$\frac{-1 + x + x^2}{1 + x^3} = -\frac{1}{1 + x} + \frac{2x^2}{1 + x^3},$$

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integrating from 0 to 1 gives (with $u = 1 + x^3$)

$$-\ln 2 + \frac{2}{3}\int_1^2 \frac{du}{u} = -\ln 2 + \frac{2}{3}\ln 2 = \boxed{-\frac{\ln 2}{3}}.$$

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Why is the swap safe at $x=1$, where $|-x^3|=1$? The first N groups are the N-th partial sum, equal to $\int_0^1\frac{(-1+x+x^2)(1-(-x^3)^N)}{1+x^3}dx$; as $|-1+x+x^2|\le 1$, the gap is at most $\int_0^1 x^{3N}dx=\frac{1}{3N+1}\to 0$.

Practice

Try the same conversion on a classic: $\sum 1/(n2^n)$.

Practice

$$\text{Compute } S = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}.$$

The $\frac{1}{n}$ is the trigger — write it as an integral, and let the 2^n join the x.

Practice (solution)

Step 1 — convert to an integral: $\frac{1}{n} = \int_0^1 x^{n-1}\,dx$; let the 2^n join the x:

$$S = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n dx.$$

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Step 2 — the inner sum is geometric: ratio $x/2 \le \frac{1}{2}$, so the swap is legal:

$$\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n = \frac{x/2}{1 - x/2} = \frac{x}{2 - x},$$

and the $\frac{1}{x}$ eats the x.

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and the $\frac{1}{x}$ eats the x.

Step 3 — integrate:

$$S = \int_0^1 \frac{dx}{2 - x} = \big[-\ln(2 - x)\big]_0^1 = \boxed{\ln 2}.$$

Answers

- BMT 2020/8: 11
- BMT 2025/5: $\pi/4$
- BMT 2024/8: $-\frac{\ln 2}{3}$
- Practice: $\ln 2$