Series–Integral Conversion

David Liu

Replace $\frac{1}{m}$ with $\int_0^1 x^{m-1}\,dx$ to turn a sum into an integral of a geometric series you can sum in closed form.

How the conversion works. The whole handout rests on one identity:

$$\frac{1}{m} = \int_0^1 x^{m-1}\,dx \qquad (m \geq 1),$$

since $\int_0^1 x^{m-1}\,dx = 1/m$. Whenever a series has a $\frac{1}{m}$-type factor, convert it:

$$\sum(\text{rest}) \cdot \frac{1}{m} = \int_0^1 \sum(\text{rest}) \cdot x^{m-1}\,dx,$$

and the inner sum is usually a geometric series that collapses in closed form. So the recipe is always the same three steps: (1) spot the $\frac{1}{m}$ factor and convert it to an integral; (2) swap sum and integral, then sum the geometric series; (3) evaluate the remaining integral.

Two series will keep appearing, so let's derive them once and for all. Differentiating $-\ln(1-x)$ gives $\frac{1}{1-x} = \sum_{a\geq 0} x^a$; integrating termwise,

$$-\ln(1-x) = \sum_{a\geq 1} \frac{x^a}{a} \qquad (|x| < 1).$$

Similarly, $\frac{d}{dt}\arctan t = \frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots$; integrating termwise from 0 to x gives $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, and at $x = 1$,

$$\frac{\pi}{4} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1},$$

the Leibniz series. (At $x = 1$ the ratio $-t^2$ reaches size 1, so check the swap with a finite sum: $\frac{1}{1+t^2} = \sum_{i=0}^{N-1}(-t^2)^i + \frac{(-t^2)^N}{1+t^2}$, and integrating over $[0,1]$ the last term contributes at most $\int_0^1 t^{2N}\,dt = \frac{1}{2N+1} \to 0$.) So the Leibniz series is rebuilt from the geometric series whenever it is needed.

One more move: a double sum whose terms factor as $A_a \cdot B_b$ splits into a product,

$$\sum_{a,b} A_a B_b = \left(\sum_a A_a\right)\left(\sum_b B_b\right),$$

which is how the two $-\ln(1-x)$ series will appear in the first example below.

Problem 1 (BMT 2020/8)**.** The summation

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{a^2b+2ab+ab^2}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Solution. The denominator factors: $a^2b+2ab+ab^2 = ab(a+b+2)$. Step 1: convert the $\frac{1}{a+b+2}$ part to an integral:

$$\frac{1}{a^2b+2ab+ab^2} = \frac{1}{ab}\int_0^1 x^{a+b+1}\,dx.$$

Step 2: sum over $a, b \geq 1$ and swap sum and integral (all terms positive). Since $\frac{x^{a+b+1}}{ab} = x\cdot\frac{x^a}{a}\cdot\frac{x^b}{b}$, the double sum factors into a product:

$$\sum_{a,b\geq 1}\frac{1}{ab(a+b+2)} = \int_0^1 x\left(\sum_{a\geq 1}\frac{x^a}{a}\right)\left(\sum_{b\geq 1}\frac{x^b}{b}\right)dx.$$

By the series fact above, each inner sum is $-\ln(1-x)$, so the sum equals

$$\int_0^1 x\ln^2(1-x)\,dx.$$

Step 3: with $u = 1-x$,

$$\int_0^1 (1-u)\ln^2 u\,du = \int_0^1 \ln^2 u\,du - \int_0^1 u\ln^2 u\,du = 2-\frac{1}{4} = \frac{7}{4},$$

the two integrals by parts. Thus $m=7$, $n=4$, and $m+n=11$. □

Problem 2 (BMT 2025/5)**.** For each real number n, let $\lfloor n\rfloor$ denote the greatest integer less than or equal to n, and let $\lceil n\rceil$ denote the least integer greater than or equal to n. Compute

$$\int_0^{\infty}\frac{(-1)^{\lfloor x\rfloor}}{\lceil x\rceil^2-\lfloor x\rfloor^2}\,dx.$$

Solution. On each open unit interval $(i, i+1)$ the floors and ceilings are constant: $\lfloor x\rfloor = i$, $\lceil x\rceil = i+1$ (the single points $x=i$ do not affect the integral). Since $(i+1)^2 - i^2 = 2i+1$,

$$\int_0^{\infty}\cdots = \sum_{i=0}^{\infty}\int_i^{i+1}\frac{(-1)^i}{2i+1}\,dx = \sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}.$$

This is the Leibniz series derived above (arctan as the integral of the geometric series $1-t^2+t^4-\cdots$, evaluated at $x=1$), so the value is $\frac{\pi}{4}$. □

Problem 3 (BMT 2024/8)**.** Evaluate

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{9n^2-12n+2}{(3n)(3n-1)(3n-2)}.$$

Solution. Partial fractions give

$$\frac{9n^2-12n+2}{(3n)(3n-1)(3n-2)} = -\frac{1}{3n-2}+\frac{1}{3n-1}+\frac{1}{3n},$$

so the series expands to

$$-1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots.$$

Write each $\frac{1}{k}$ as $\int_0^1 x^{k-1}\,dx$ and regroup by powers of x:

$$\int_0^1 \Big[(-1+x+x^2)+(x^3-x^4-x^5)+(-x^6+x^7+x^8)+\cdots\Big]\,dx.$$

The bracketed terms form a geometric series with ratio $-x^3$, summing to $\frac{-1+x+x^2}{1+x^3}$. (The swap is safe even though $|-x^3|$ reaches 1 at $x=1$: the first N groups are exactly the N-th partial sum of the series, and they equal $\int_0^1 \frac{(-1+x+x^2)(1-(-x^3)^N)}{1+x^3}\,dx$; since $|-1+x+x^2|\le 1$ on $[0,1]$, the difference from the full integral is at most $\int_0^1 x^{3N}\,dx = \frac{1}{3N+1}\to 0$.) Since

$$\frac{-1+x+x^2}{1+x^3} = -\frac{1}{1+x}+\frac{2x^2}{1+x^3},$$

integrating from 0 to 1 gives

$$-\ln 2+\frac{2}{3}\int_1^2 \frac{du}{u} = -\ln 2+\frac{2}{3}\ln 2 = -\frac{\ln 2}{3},$$

with $u=1+x^3$. □

Exercise 4. Compute $\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$ by writing $\frac{1}{n}$ as an integral. Hint: 1

Answers: 11; $\pi/4$; $-\frac{\ln 2}{3}$; $\ln 2$.

Hints

1. Use $\frac{1}{n} = \int_0^1 x^{n-1}\,dx$, let the 2^n join the x, swap sum and integral, and sum the resulting geometric series.